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Sharp Interface Motion of a Binary Fluid Mixture

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We derive hydrodynamic equations describing the evolution of a binary fluid segregated into two regions, each rich in one species, which are separated (on the macroscopic scale) by a sharp interface. Our starting point is a Vlasov-Boltzmann (VB) equation describing the evolution of the one particle position and velocity distributions, $f_i(x, v, t), i = 1, 2$. The solution of the VB equation is developed in a Hilbert expansion appropriate for this system. This yields incompressible Navier-Stokes equations for the velocity field u and a jump boundary condition for the pressure across the interface. The interface, in turn, moves with a velocity given by the normal component of u.

KEY WORDS: Interface evolution, Navier-Stokes equations, Binary fluids, Phase segregation.

1. INTRODUCTION

When a fluid mixture is suddenly quenched from a homogeneous equilibrium state into an immiscible region of the phase diagram, where the uniform state is thermodynamically unstable, it evolves to a new state consisting of two coexisting phases, each rich in one species. There are various stages of this phase segregation process. It starts with the formation of very diffuse interfaces that sharpen with time and then move slowly, driven by surface tension effects.[23,26] In a series of papers,[2,3,9] we have investigated a kinetic model of this phenomenon. Our model system consists of two species of particles interacting via a repulsive long range potential between different species and a short range hard core between all particles. At low temperature the system will undergo phase segregation.

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The evolution of the system, in a space region Ω which we assume for simplicity to be a three-dimensional torus of volume L^3, is given, on the kinetic space time scale, by two coupled Vlasov-Boltzmann (VB) equations for the one-particle distributions $f_i(x, v, \tau)$, $i = 1, 2$. The Vlasov term models the long range repulsive interaction between different species while the Boltzmann kernel takes account of the hard collisions. These equations, which conserve the mass of each species, the total momentum and energy, have the form:

$$\partial_\tau f_i + v \cdot \nabla_x f_i + F_i \cdot \nabla_v f_i = J(f_i, f_1 + f_2), \quad i = 1, 2 \tag{1.1}$$

where $J(f, g)$ is the (non symmetric) Boltzmann collision operator for hard spheres,[11]

$$J(f, g) = \int_{\mathbb{R}^3} dv_* \int_{S_2} d\omega B(|v - v_*|, \omega)[f(v')g(v'_*) - f(v)g(v_*)]$$

where S_2 is the $2 - d$ sphere in $\mathbb{R}^3$, $d\omega$ is the surface measure on it, the vectors v, v_* are the outgoing velocities of a binary elastic collision between two particles with incoming velocities v' and v'_* and $B(|v - v_*|, \omega) = \frac{1}{2}|(v - v_*) \cdot \omega|$.

Moreover, $F_i(x, \tau) = -\nabla_x V_i(x, \tau)$ are potential force terms with V_i the self-consistent Vlasov potentials

$$V_i(x, \tau) = \int_\Omega U_\ell(|x - x'|)\rho_j(x', \tau)dx', \quad i \neq j, \tag{1.2}$$

$\rho_j(x, \tau) = \int_\Omega f_j(x, v, \tau)dv$ are the mass densities and U_ℓ is the long range interaction potential with range ℓ. We write it as

$$U_\ell(x) = \ell^{-3} U\left(\frac{x}{\ell}\right) \tag{1.3}$$

with $U(x)$ a non negative smooth compactly supported function whose integral is normalized to unity,

$$\int_{\mathbb{R}^3} dx U(x) = 1.$$

This fixes the energy scale of the system. In (1.2) and (1.3) we have made explicit the dependence on the range ℓ of the potential for future convenience. We also note that (1.1) holds on the space scale where the mean free path λ is of the order of unity: we fix our units so that $\lambda = 1$.

Computer simulations show that at sufficiently low temperature the solutions to (1.1) relax to non homogeneous equilibrium states.[4] To characterize these equilibrium states it is useful to consider the (negative of the) entropy functional

$$\mathcal{H}(f_1, f_2) = \sum_{i=1}^{2} \int dx dv f_i \log f_i, \tag{1.4}$$

which is a Liapunov functional for (1.1) in the sense that

$$\frac{d}{dt}\mathcal{H}(f_1, f_2) \leq 0 \tag{1.5}$$

The time independent solutions of (1.1) then satisfy the equality in (1.5). They are Maxwellian's with constant mean value $u = 0$, which can be set equal to zero, variance $T = \beta^{-1}$, and possibly nonuniform densities $\rho_i = \int dv f_i(x, v, \tau)$ satisfying

$$T \log \rho_i(x) + \int \mathrm{d}x' U_\ell(|x - x'|)\rho_j(x') = C_i, \quad i = 1, 2, i \neq j. \tag{1.6}$$

We shall call these states equilibrium states.

An alternative way to obtain equilibrium states is to minimize the entropy functional under the constraints on the total energy and total masses. The densities will then be determined[9] as the minimizers of the free energy functional

$$\mathcal{F}(\rho_1, \rho_2) = \int \mathrm{d}x\, [(\rho_1 \log \rho_1) + (\rho_2 \log \rho_2)] + \beta \int \mathrm{d}x\, \mathrm{d}y U_\ell(|x - y|)\rho_1(x)\rho_2(y). \tag{1.7}$$

Equation (1.6) are indeed the Euler-Lagrange equations for this minimization problem. T is determined by the constraint on the energy and C_i by the constraints on the masses.

It is proved in Ref. (9), under the assumption of a monotone $U_\ell(x)$, that at low temperature there are non homogeneous solutions to (1.6), thermodynamically stable in the sense that they minimize $\mathcal{F}$.

On the infinite line non homogeneous minimizers $w_i(z)$, $z \in \mathbb{R}$ and $i = 1, 2$, of the excess free energy arise when prescribing asymptotic values for the densities corresponding to the two different phases coexisting at equilibrium: such minimizers, called *fronts*, have monotonicity properties and interpolate smoothly, over a region of size ℓ between the asymptotic constant values. Similar results hold in higher dimensions in the sense that, for $L \gg \ell$, the volume Ω is divided into two regions where the densities are those of the pure phases with an interpolating region called the *interface* whose size is again of order ℓ.

In Ref. (2) we studied the time dependent macroscopic behavior of this system by introducing a scale separation parameter ε between the kinetic length scale λ and the typical macroscopic one L, while keeping the range of the potential ℓ macroscopic. We derived there, via a rigorous Chapmann-Enskog expansion, Vlasov-Navier-Stokes (VNS) equations which describe the behavior of the system on length scales of order ε^{-1} and time scales of order ε^{-1} for small ε in a situation in which the interface is diffuse. In these VNS equations there are, beyond the usual terms present in the compressible Navier-Stokes equations, diffusive terms coming from the presence of the self-consistent force. In particular, the equation for the concentration can be put in the form of a gradient flux of an energy functional which is similar to an exact evolution equation (analogous to the Cahn-Hilliard equation[8]) derived for a stochastic lattice model of a binary alloy.[17] In the limit $\varepsilon \to 0$, the VNS equations reduce to the Vlasov-Euler (VE) equations. Both VE and VNS have non trivial stationary solutions with the same solitonic profile as in the VB equation.

In Ref. (3) we started to study the late stages of the segregation process in which the interface becomes sharp. We argued there that the interface should move in its normal direction following the incompressible velocity field solution of the Navier-Stokes equation, while the pressure should satisfy Laplace's law relating it to the surface tension and curvature. Numerical simulations confirm this scenario. This limiting evolution of the system is ruled by the following free boundary problem. Let Γ^0 be a regular surface in a $3-d$ torus Ω (the interface at time zero) dividing the torus in two regions Ω^+ and Ω^-. For each t one has to find a surface Γ_t, moving with velocity V, a continuous velocity field $u(\cdot, t)$ and a pressure function $p(\cdot, t)$ such that

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \eta \Delta u \\ V = -u \cdot \nu \\ [p]^+_- = K_{\Gamma_t} \sigma \qquad \text{on } \Gamma_t \\ \operatorname{div} u = 0 \\ \Gamma_0 = \Gamma^0, \quad u(\cdot, 0) = u_0(\cdot). \end{cases} \tag{1.8}$$

were η is the kinematic viscosity, σ denotes surface tension, $\nu(\cdot, t)$ is the normal to the surface pointing towards Ω^+, K_{Γ_t} stands for the curvature of Γ_t and $[h]^+_- = h^+ - h^-$ stands for the jump of the observable h across Γ_t. We note that the signs in the second and third of the Eq. (1.8) depend on the conventions on V and ν specified in Section 2, where we shall give the precise definitions.

This free boundary problem was first formulated to describe the oscillations of an impermeable interface separating two viscous fluids in Refs. (18, 20). Chandrasekar[12] then studied the linear stability of this system. We mention that recently[14,25] have obtained existence uniqueness and regularity results for the one-side case, namely the flow of an incompressible Navier-Stokes fluid confined in a region with free boundary where suitable surface tension boundary conditions are specified. Such results can be extended to the present two-side case.[24]

To understand the intrinsic dynamics described by (1.8) let us give some basic properties of the motion:

(1) volume conservation of each phase

$$\frac{d}{dt}|\Omega^+| = 0$$

(2) basic energy identity

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega dx|u(x,t)|^2 + \sigma|\Gamma_t|\right] = -\eta \int_\Omega dx|\nabla u(x,t)^2$$

These properties show that this flow diminishes the area of the boundary while conserving the volume and at the same time forces the velocity field to decay to zero. Hence, the stationary solution should be characterized by $u = 0$ and a surface Γ determined by the isoperimetric problem on the torus, separating Ω in two phase regions with different values of the pressure such that $[p]_-^+ = K_\Gamma \sigma$.

Here we want to derive these equations for the motion of a sharp interface by a Hilbert expansion of the solution of (1.1). We are interested in hydrodynamical flows in the macroscopic regime obtained by sending ε, the ratio between the mean free path λ and $L = |\Omega|^{1/3}$ to zero. (Note that in units in which $\lambda = 1, \varepsilon = L^{-1}$). In addition, for reasons that will become clear below, we want to consider a situation in which the interface is sharp already on the kinetic scale in the sense that its thickness is much smaller than the mean free path. Since the width of the interface on the kinetic scale is of order ℓ, we want to have ℓ much smaller than L. It turns out that the right choice is $\ell = \varepsilon\lambda = \varepsilon$.

It should be noted that the formal derivation of (1.1) presented in Ref. (2), from an underlying microscopic model, makes sense in this scaling. The microscopic interaction range of U can be chosen small compared to the mean free path, yet sufficiently large so that the number of particles within the interaction range grows indefinitely in the limit we consider.

We introduce the macroscopic coordinates $r = \varepsilon x$. Since $x \in \Omega = \mathbb{T}_L$, the torus of size L, $r \in \mathbb{T}_1$, the torus of size 1. We wish to study the small ε behavior of a solution of the VB equations. In order to observe diffusive effects one has to consider very long times, i.e. set $\tau = \varepsilon^{-2}t$, with t the macroscopic time. Setting $f_i^\varepsilon(r, v, t) = f_i(\varepsilon^{-1}r, v, \varepsilon^{-2}t)$, $\rho_i^\varepsilon(r, t) = \int dv f_i^\varepsilon(r, v, t)$, the Vlasov-Boltzmann equation, in this space-time scaling, becomes

$$\partial_t f_i^\varepsilon + \varepsilon^{-1} v \cdot \nabla_r f_i^\varepsilon + \varepsilon^{-1} F_i^\varepsilon \cdot \nabla_v f_i^\varepsilon = \varepsilon^{-2} J\big(f_i^\varepsilon, f_1^\varepsilon + f_2^\varepsilon\big) \tag{1.9}$$

$$F_i^\varepsilon(r, t) = -\nabla_x \int_{\mathbb{T}_1} \mathrm{d}r' \varepsilon^{-6} U(\varepsilon^{-2}|r - r'|) \int_{\mathbb{T}_1} \mathrm{d}v' f_j^\varepsilon(r', v', t). \tag{1.10}$$

$$=: -\nabla_r U^\varepsilon * \rho_j^\varepsilon$$

Consider now a situation in which there is, at the initial time, an interface separating the system in two regions with densities corresponding to the equilibrium values at temperature T (i.e. with coexistence of two phases, one richer in species 1 and the other richer in species 2). For ε finite, we approximate the density profiles by one-dimensional fronts in the direction orthogonal to the interface in each point. The fronts interpolate between the two phases on a scale ε^2. If the interface were flat, this would be a stationary solutions of the VB equations. Since the interface is not flat the fluid starts to move because of the unbalance of the pressure on the two sides of the interface (surface tension). This pushes the interface to move with the component of the fluid velocity in the direction of the normal at each point of the surface. Since the initial density in the bulk is the equilibrium and the space-time scaling is diffusive, we expect that the fluid in the bulk will evolve as an incompressible Navier-Stokes (INS) fluid. We recall that the INS equations can be obtained from an equation of the type (1.9) when the average velocity is small (low Mach numbers). Therefore, the effect of the surface tension has to be suitably small in order not to get too big velocities. The surface tension effect is proportional to the size of the interface and this is the reason why we choose $\ell = \varepsilon$.

In Section 2, after introducing some notations, we construct an expansion in the bulk (outer) and a different expansion close to the interface (inner) and impose matching conditions on an intermediate region. In Section 3 we examine explicitly the first and second order terms in this expansion. At the first order we find the following free boundary problem for the velocity field u:

$$\partial_t u + (u \cdot \nabla)u + \nabla p = \eta \Delta u \tag{1.11}$$

In (1.11) the kinematic viscosity η is obtained from the Boltzmann equation as in,[11] u is continuous across the interface Γ_t whose normal velocity is given by

$$v_{\Gamma_t}(r) = -u(r,t) \cdot \nu(r,t) \tag{1.12}$$

while the pressure is discontinuous at the surface and satisfies Laplace's law

$$(p_+ - p_-) = \sigma K, \tag{1.13}$$

Here (p_+) p_- is the pressure on the side of Γ_t (not) containing the normal ν; K is the mean curvature of Γ_t, σ is the surface tension given in terms of one-dimensional fronts w_i as

$$\sigma = \frac{1}{2} \int_{\mathbb{R}} (z' - z) \sum_{i \neq j = 1,2} \frac{dw_i(z)}{dz} \tilde{U}(z - z') w_j(z') dz dz' \tag{1.14}$$

and $\tilde{U}$ is obtained from U by integrating out the two variables, parallel to the surface. Moreover, we get equations for the first correction to the temperature $T^{(1)}$ (which at order zero is the constant $\bar{T}$) and concentration $\phi^{(1)}$ which are similar to phase field equations[7]

$$D_t \phi^{(1)} = DA_1 \Delta \phi^{(1)} + DA_2 \Delta T^{(1)} \tag{1.15}$$

$$D_t T^{(1)} = k \Delta T^{(1)} + A_3 D_t \phi^{(1)} \tag{1.16}$$

where the diffusion coefficient D, the heat conductivity k and the constants A_i are explicit functions of $\bar{\rho}$, $\bar{T}$.[2]

In Section 4 we show that the second order corrections to the hydrodynamic fields solve a boundary value problem on a prescribed surface, which is the interface determined at the first order. We obtain also a correction to the velocity of the interface.

Finally, in Section 5 we consider the VNS equations,[2] which one obtains from the Vlasov-Boltzmann equations when scaling space and time by a factor ε^{-1} (hyperbolically), while keeping ℓ, the range of U_ℓ, finite on the macroscopic scale. For the VNS equations we study the sharp interface limit by matching expansions. This is the strict analog of what it is usually done in the literature in the case of alloys, starting from the Cahn-Hilliard equation[8] (or equivalent non local equations[17]) and deriving the motion of the sharp interface (e.g. Mullins-Sekerka motion[22]). The difference is that we have to consider a limit in which the interface is very sharp, since we have to scale its width by a factor ε^2, to be compared with the factor ε for the alloys and to send to zero the Mach number, to get the right hydrodynamics in the bulk. Surprisingly or not, we get the same free boundary problem (1.8) as the one in Section 3.

2. HILBERT SERIES

We start with a system in which an interface is present. Since the stationary non homogeneous solution of (1.9) is given by the Maxwellian multiplied by the front density profiles we let our system start initially close to that stationary solution. More precisely, we choose as initial density $f_i^\varepsilon(r, v) = M(v)\rho_i^{(\varepsilon)}$, where $M(v)$ is the Maxwellian with unit mass, zero mean velocity and temperature $\bar{T}$,

$$M(v) = (2\pi \bar{T})^{-3/2} \exp[-v^2/2\bar{T}]. \tag{2.1}$$

The density profiles $\rho_i^{(\varepsilon)}$ are very close to a profile such that in the bulk its values are $\rho_i^\pm$, the values of the densities in the two pure phases at temperature $\bar{T}$, and the interpolation between them, at the interface, is realized along the normal direction in each point by the fronts. By the symmetry properties of our model, $\rho_2^\mp = \rho_1^\pm$.

Consider a smooth surface $\Gamma^0 \subset \Omega$, and let $d(r, \Gamma^0)$ be the signed distance of the point $r \in \Omega$ from the interface. We assume an initial profile for the densities $\rho_i^{(\varepsilon)}$ of the following type: at distance greater than $O(\varepsilon^2)$ from the interface (in the bulk) the density profiles $\rho_i^{(\varepsilon)}(r)$ are almost equal to $\rho_i^\pm$; at distance $O(\varepsilon^2)$ (near the interface) we choose

$$\rho_i^{(\varepsilon)}(r) = w_i(\varepsilon^{-2} d(r, \Gamma^0)) + O(\varepsilon^2). \tag{2.2}$$

Here, for any smooth surface Γ separating the 1-richer phase from the 2- richer phase, $d(r, \Gamma)$ denotes the signed distance from r to Γ taken positive in the 1-richer region. Moreover $w_i(z)$ are the so called *front* (or sometimes *soliton*) solutions with asymptotic values $\rho_i^\pm$. These are the one dimensional stationary solutions of

$$\bar{T} \log w_i(q) + \int_{\mathbb{R}} dq' \tilde{U}(|q - q'|) w_j(q') = C_i \tag{2.3}$$

where $\tilde{U}(q) = \int_{\mathbb{R}^2} dy U(\sqrt{q^2 + |y|^2})$ and C_i are constants determined by the conditions at infinity $\rho_i^{\pm}$. We will choose values of $\rho_i^{\pm}$ and temperature $\bar{T}$, corresponding to the phase transition region, which implies $C_1 = C_2$. Existence of such solutions can be proved along the same lines of the proofs in,[15] and details will be presented elsewhere. Since these solutions are unique up to a translation we fix a solution by imposing that $w_1(0) = w_2(0)$. Among the relevant properties of such equations, we will use their exponential decay to the asymptotic values which also follows from the arguments in Ref. (15).

Let Γ_t^ε be an interface at time t defined by

$$\Gamma_t^\varepsilon = \{r \in \Omega : \rho_1^\varepsilon(r, t) = \rho_2^\varepsilon(r, t)\}$$

where $\rho_i^\varepsilon(r, t) = \int dv f_i^\varepsilon(r, v, t)$. Let $t_0 > 0$ be such that Γ_t^ε is regular for $t \in [0, t_0]$. Let $d^\varepsilon(r, t)$ be the signed distance of $r \in \Omega$ from the interface Γ_t^ε, such that $d^\varepsilon > 0$ in $\Omega_t^{\varepsilon,+}$ and $d^\varepsilon < 0$ in $\Omega_t^{\varepsilon,-}$, where $\Omega = \Gamma_t^\varepsilon \cup \Omega_t^{\varepsilon,+} \cup \Omega_t^{\varepsilon,-}$. Hence, by continuity, $\Omega_t^{\varepsilon,+}$ is the 1-richer region and $\Omega_t^{\varepsilon,-}$ is the 2-richer region. For sake of simplicity we drop from now on the superscript ε. For any r such that $|d(r, t)| < \frac{1}{\delta}$ defined below, there exists $s(r) \in \Gamma_t$ such that

$$\nu(s(r))d(r, t) + s(r) = r$$

where $\nu(s(r))$ is the normal to the surface Γ_t in $s(r)$ pointing toward $\Omega_t^{\varepsilon,+}$, namely

$$\nu(s(r)) = \nabla_r d(r, t).$$

Define the normal velocity of the interface as

$$V(s(r)) = \partial_t d(r, t).$$

The curvature K (the sum of the principal curvatures) is given by $K = \Delta_r d(r, t)$. Define, for ε_0 small enough,

$$\mathcal{N}(\delta) := \{r : |d(r, t)| < \delta\}$$

where

$$\frac{1}{\delta} = \max_{t \in [0, t_0], \varepsilon < \varepsilon_0} k(\Gamma_t), \quad \text{and} \quad k(\Gamma_t) = \sup_{r \in \Gamma_t} k(r).$$

$k(r)$ being the maximum of the principal curvatures in $s(r)$

We now try to construct an approximate solution to the rescaled VB Eq. (1.9). We follow the approach based on truncated Hilbert expansions introduced by Caflisch.[6] This method has been improved by including boundary layer expansions in Ref. (16), where it is proved the hydrodynamic limit for the Boltzmann equation in a slab. Here we try to adapt the arguments in Ref. (16) to the fact that the boundary is not given a priori and has to be found as a result of the expansion. We will do it by using the method of matching expansions. However, the estimate on the remainder (see (2.12) below), which is crucial to establish the rigorous validity of our expansion, will not be discussed in this paper. For an alternative approach, in the context of the Boltzmann equation, see Ref. (28). The Hilbert expansion is nothing but a formal power expansion in ε for the solution of the kinetic equations

$$f_i^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n f_i^{(n)}. \tag{2.4}$$

Since we expect the behavior of the solution be different in the bulk and near the interface we decompose $f_i^{(n)}$ in two parts: the bulk part $\hat{f}_i^{(n)}$ and boundary layer part $\tilde{f}_i^{(n)}$. The latter will be fast varying functions close to the interface different from zero only in $\mathcal{N}(\delta)$, namely they depend on (r, t) in the following way

$$\tilde{f}_i^{(n)} = \tilde{f}_i^{(n)}(\varepsilon^{-2} d(r, t), r, t)$$

More precisely, a fast varying function $h(r, t)$ for $r \in \mathcal{N}(\delta)$ can be represented as a function $h(z, r, t)$, $z = \varepsilon^{-2} d(r, t)$, with the condition $h(z, r + \ell\nu(s(r)), t) = h(z, r, t)$, ℓ small enough. Hence we can write

$$\begin{aligned} \nabla_r h &= \frac{1}{\varepsilon^2}\nu\partial_z h + \overline{\nabla}_r h; \quad \partial_t h = \frac{1}{\varepsilon^2} V\partial_z h + \overline{\partial}_t h; \\ \Delta_r h &= \frac{1}{\varepsilon^4}\partial_z^2 h + \frac{1}{\varepsilon^2}(\nabla_r \cdot \nu)\partial_z h + \overline{\Delta}_r h \end{aligned} \tag{2.5}$$

where the bar on the derivative operators means derivatives with respect to r or t, with the other variables fixed. Note that $\nu \cdot \overline{\nabla}_r h(z, r, t) = 0$.

We shall use the notation $\rho_i^\varepsilon = \int dv f_i^\varepsilon$, and we denote by $\hat{h}$ (resp. $\tilde{h}$) a function $h(f_i^{(n)})$ whenever is evaluated on $\hat{f}_i^{(n)}$ (resp. $\tilde{f}_i^{(n)}$).

Since the surface Γ_t^ε depends on ε, so does $d(r, t)$. Therefore we expand also the signed distance:

$$d(r, t) = \sum_{i=0}^{\infty} \varepsilon^n d^{(n)}(r, t). \tag{2.6}$$

The condition $|\nabla_r d|^2 = 1$ is equivalent to:

$$\begin{aligned} |\nabla_r d^{(0)}|^2 &= 1, \quad \nabla_r d^{(0)} \nabla_r d^{(1)} = 0, \\ \nabla_r d^{(0)} \nabla_r d^{(j)} &= -\frac{1}{2}\sum_{i=1}^{j-1} \nabla_r d^{(i)} \nabla_r d^{(j-1)}, \quad j \geq 2 \end{aligned}$$

so that $d^{(0)}$ can be interpreted as a signed distance from an interface that we denote by $\bar{\Gamma}_t$. As a consequence of (2.6) the velocity of the interface Γ_t has the form

$$\sum_{i=0}^{\infty} \varepsilon^i V^{(i)}, \quad V^0 := \bar{V}.$$

We remark that velocity V determines the surface evolving with it. The velocity $\bar{V}$ will generate an "order zero" interface $\bar{\Gamma}_t$. The interface generated by $\sum_i \varepsilon^i V^{(i)}$ will be a deformation, small for small ε, of $\bar{\Gamma}_t$.

Our aim is to give an algorithm to construct the terms of the expansion at any order. Below we just compute the first few orders and indicate how to continue.

We replace (2.4) in the equations and equate terms of the same order in ε. First, we need to compute the terms in the expansion for the force term $F_i^\varepsilon = \sum_{n=0}^N \varepsilon^n F_i^{(n)}$. To write the expansion for the force term F_i^ε we expand $U^\varepsilon * \sum_{n=0}^\infty \rho_j^n = \sum_{n=0}^\infty \varepsilon^n g_i^{(n)}$ and $F_i^{(n)} = -\nabla_r g_i^{(n)}$. We have to be careful in dealing with the terms involving the potential U^ε. These terms are computed in Appendix A.

We define

$$\mathcal{N}^0(m) := \{r : |d^{(0)}(r,t)| < m\}, \bar{\Gamma}_t := \{r : |d^{(0)}(r,t)| = 0\}, \Omega^\pm := \{r : |d^{(0)}(r,t)| > 0(< 0)\}$$

and fix m so that $\mathcal{N}^0(m) \subset \mathcal{N}(\delta)$.

We assume that in $\Omega^\pm \setminus \mathcal{N}^0(m)$

$$f_i^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \hat{f}_i^{(n)}. \tag{2.7}$$

and that in $\mathcal{N}^0(m)$ the solution is of the form

$$f_i^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \tilde{f}_i^{(n)} \tag{2.8}$$

We will match the inner and outer expansions in $z = \varepsilon^{-1} c$ with $c = \varepsilon^\alpha$, $\alpha \in (0, 1)$. Hence, we require as $z \to \pm\infty$ (see Ref. (7))

$$\begin{aligned}
\tilde{f}_i^{(0)} &= \big(\hat{f}_i^{(0)}\big)^\pm + O(e^{-\alpha|z|})\\
\tilde{f}_i^{(1)} &= \big(\hat{f}_i^{(1)}\big)^\pm + O(e^{-\alpha|z|})\\
\tilde{f}_i^{(2)} &= \big(\hat{f}_i^{(2)}\big)^\pm + \nu^{(0)} \cdot (\nabla_r \hat{f}_i^{(1)})^\pm d^{(1)} + O(e^{-\alpha|z|}) \qquad (2.9)\\
\tilde{f}_i^{(3)} &= \big(\hat{f}_i^{(3)}\big)^\pm + \nu^{(0)} \cdot \big(\nabla_r \hat{f}_i^{(2)}\big)^\pm d^{(1)} + \big(\nabla_r \hat{f}_i^{(1)}\big)^\pm \cdot \big(\nu^{(0)}\big(z - d^{(2)}\big) + \nu^{(1)} d^{(1)}\big)
\end{aligned}$$

$$+\frac{1}{2}\big(\partial_{r_h}\partial_{r_k}\hat{f}_i^{(1)}\big)^{\pm}\nu_h^{(0)}\nu_k^{(0)}\big(z-\big(d^{(1)}\big)^2\big)+O(e^{-\alpha|z|})$$
$$\dots$$

where, for a slow function $\hat{h}$, the symbol $(\hat{h})^{\pm}$ stands for $\lim_{\ell\to 0^{\pm}}\hat{h}(r+\nu\ell), r\in\bar{\Gamma}_t$, and the same for the derivatives. We denote by $\nu^{(j)}$ the vectors $\nabla_r d^{(j)}$ but it is important to keep in mind that they are not unit vectors (except $\nu^{(0)}$). To lighten the formulas we have omitted the terms involving derivatives of $f_i^{(0)}$ since they will turn out to be zero.

We plug $f^{\varepsilon}=\sum_{n=0}^{\infty}\varepsilon^n f^{(n)}$ in the rescaled equations and write a set of equations in $\Omega\backslash\mathcal{N}^0(m)$ involving only functions $\hat{f}_i^{(n)}$ as well in $\mathcal{N}^0(m)$ for $\tilde{f}_i^{(n)}$ by equating order by order in ε. We use the notation:

$$\big(U^{\varepsilon}*\hat{\rho}_j^{\varepsilon}\big)(r,t)=\sum_{n=0}^{N}\varepsilon^n\hat{g}_i^{(n)},\qquad \big(U^{\varepsilon}*\tilde{\rho}_j^{\varepsilon}\big)(r,t)=\sum_{n=0}^{N}\varepsilon^n\tilde{g}_i^{(n)}$$

The expressions for $\hat{g}_i^{(n)}$ and $\tilde{g}_i^{(n)}$ are given in Appendix A, formulas (A.8) and (A.9). We adopt the convention that $\hat{f}_i^{(n)}$ and $\tilde{f}_i^{(n)}$ vanish for $n<0$.

Outer expansion
In $\Omega^{\pm}\setminus\mathcal{N}^0(m)$ for $n\geq 0$:

$$\begin{aligned}\partial_t\hat{f}_i^{(n-2)}+v\cdot\nabla_r\hat{f}_i^{(n-1)}+\sum_{l,l'\geq 0:l+l'=n-1}\hat{F}_i^{(l)}\cdot\nabla_v\hat{f}_i^{(l')}\\=\sum_{k,k':k+k'=n}J\big(\hat{f}_i^{(k)},\hat{f}_2^{(k')}+\hat{f}_1^{(k')}\big).\end{aligned}\tag{2.10}$$

Inner expansion
In $\mathcal{N}^0(m)$ for $n0$:

$$\begin{aligned}&\sum_{l,l'\geq 0:l+l'=n-1}V^{\ell'}\partial_z\tilde{f}_i^{(l)}+\sum_{k+k'=n}\nu^{(k)}\cdot v\partial_z\tilde{f}_i^{(k')}+v\cdot\overline{\nabla}_r\tilde{f}_i^{(n-2)}+\overline{\partial}_t\tilde{f}_i^{(n-3)}\\&-\sum_{l,l'\geq 0:l+l'=n}\left[\sum_{k+k'=\ell}\nu^{(k)}\,\partial_z\tilde{g}_i^{(k')}+\overline{\nabla}_r\tilde{g}_i^{(l-2)}\right]\cdot\nabla_v\tilde{f}_i^{(l')}\\&=\sum_{k,k':k+k'=n-1}J\big(\tilde{f}_i^{(k)},\tilde{f}_1^{(k')}+\tilde{f}_2^{(k')}\big).\end{aligned}\tag{2.11}$$

The strategy for a rigorous proof is to construct, once the functions $f_i^{(n)}$ have been determined, the solution in terms of a truncated Hilbert expansion as

$$f_i^{\varepsilon}=\sum_{n=0}^{N}\varepsilon^n f_i^{(n)}+\varepsilon^m R_i\tag{2.12}$$

where the functions are evaluated in $z = \varepsilon^{-2} d^N(r,t)$, with $d^N(r,t) = \sum_{n=0}^{N-2} \varepsilon^n d^{(n)}(r,t)$ and then write a weakly non linear equation for the remainder. In this approach it is essential to have enough smoothness for the terms of the expansion. On the contrary, they would be discontinuous on the border of $\mathcal{N}^0(m)$ since $\tilde{f}^{(n)}$ are not exactly equal to $\hat{f}^{(n)}$ there, but differ for terms exponentially small in ε. One can modify the expansion terms by interpolating in a smooth way between the outside and the inside getting smooth terms which do not satisfy the equations for terms exponentially small in ε, that can be included in the remainder. With this in mind, we did not put in the equations the terms coming from the force such that in the convolution r is in $\mathcal{N}^0(m)$ and r' in $\Omega \setminus \mathcal{N}^0(m)$. That is possible because the potential is of finite range.

3. BUILDING EXPANSION TERMS

We start examining the bulk equations order by order.

Outer expansion

The Eq. (2.10) reads for $n = 0$ as

$$J\big(\hat{f}_i^{(0)}, \big(\hat{f}_1^{(0)} + \hat{f}_2^{(0)}\big)\big) = 0$$

which implies[13] that $\hat{f}_i^{(0)}$ are Maxwellians, with the same temperature and mean velocity, as far as the dependence on v is concerned. We choose $\hat{\rho}_i^{(0)}$, $\hat{T}^{(0)}$ and $\hat{u}^{(0)}$ to be constant in $\Omega^\pm \setminus \mathcal{N}^0(m)$, equal to $\bar{\rho}_i^\pm$, $\bar{T}$ and 0 so to match the initial datum. Therefore, we put $\hat{f}_i^{(0)}(r,v,t) = \hat{\rho}_i^{(0)} M$ (see (2.1). We denote by $\bar{\rho}_i(r,t)$ the functions

$$\bar{\rho}_1(r,t) = \bar{\rho}^+ \chi_+(r) + \bar{\rho}^- \chi_-(r); \quad \bar{\rho}_2(r,t) = \bar{\rho}^- \chi_+(r) + \bar{\rho}^+ \chi_-(r)$$

where $\chi_\pm(r)$ are the (smoothed) characteristic functions of $\Omega^\pm \setminus \mathcal{N}^0(m)$ and $\bar{\rho}^\pm = \rho_1^\pm$. We observe that the total density at order zero $\bar{\rho}(r,t) = \bar{\rho}_1(r,t) + \bar{\rho}_2(r,t) = \bar{\rho}^+ + \bar{\rho}^-$ has the same value in $\Omega^+ \setminus \mathcal{N}^0(m)$ and $\Omega^- \setminus \mathcal{N}^0(m)$ while the concentration $\bar{\phi}(r,t) = \bar{\rho}_1(r,t) - \bar{\rho}_2(r,t)$ changes sign. Moreover, the pressure at order zero, $\hat{P}^{(0)}(r,t) = \hat{\rho}^{(0)} \hat{T}^{(0)}$ has a constant value since depends on the total density.

For $n = 1$ we have in (2.10)

$$v \cdot \nabla_r \hat{f}_i^{(0)} - \nabla_r \hat{g}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(0)} = J\big(\hat{f}_i^{(1)}, \hat{f}_1^{(0)} + \hat{f}_2^{(0)}\big) + J\big(\hat{f}_i^{(0)}, \hat{f}_1^{(1)} + \hat{f}_2^{(1)}\big)$$

which implies

$$J\big(\hat{f}_1^{(0)} + \hat{f}_2^{(0)}, \hat{f}_1^{(1)} + \hat{f}_2^{(1)}\big) = 0$$

In terms of the sum and the difference defined as

$$\hat{\gamma}_{\pm}^{(k)} = \hat{f}_1^{(k)} \pm \hat{f}_2^{(k)}$$

the previous conditions reads as

$$\mathcal{L}\hat{\gamma}_{+}^{(1)} = 0, \quad \mathcal{Z}\hat{\gamma}_{-}^{(1)} = -J\big(\bar{\phi}M, \hat{\gamma}_{+}^{(1)}\big) \tag{3.1}$$

where we have used that $\hat{f}^{(0)}$ is Maxwellian in velocity and constant in space. Moreover

$$\mathcal{L}h := J(\bar{\rho}M, h) + J(h, \bar{\rho}M), \quad \mathcal{Z}h = J(h, \bar{\rho}M) \tag{3.2}$$

are the standard linearized Boltzmann operator and the linear Boltzmann operator respectively.

The first of conditions (3.1) implies

$$\gamma_{+}^{(1)}(r, v, t) = \bar{\rho}M(v)\left(\hat{\rho}^{(1)} + \hat{u}\cdot\frac{v}{\bar{T}} + \hat{T}^{(1)}\frac{v^2 - 3\bar{T}}{2\bar{T}^2}\right) := \bar{\rho}M\sum_{\ell=0}^{4}\chi_\ell(v)\alpha_\ell^{(1)}(x, t), \tag{3.3}$$

and the second one implies

$$\mathcal{Z}\left[\hat{\gamma}_{-}^{(1)} - \frac{\bar{\phi}}{\bar{\rho}}\hat{\gamma}_{+}^{(1)}\right] = -\frac{\bar{\phi}}{\bar{\rho}}\mathcal{L}\hat{\gamma}_{+}^{(1)} = 0$$

Here, $\chi_0 = 1$, $\chi_i = v_i$, $\chi_4 = \frac{1}{2}|v|^2$. Since the null space of $\mathcal{Z}$ is given by multiples of M, $\hat{\gamma}_{-}^{(1)} - \frac{\bar{\phi}}{\bar{\rho}}\hat{\gamma}_{+}^{(1)}$ has to be equal to M times a function of (r, t). We introduce the functions $\psi^{(1)}$ and $\hat{\phi}^{(1)}$ as

$$\hat{\gamma}_{-}^{(1)} - \frac{\bar{\phi}}{\bar{\rho}}\hat{\gamma}_{+}^{(1)} = \psi^{(1)}\bar{\rho}M; \quad \hat{\phi}^{(1)} := \int dv\hat{\gamma}_{-}^{(1)} = \bar{\rho}\psi^{(1)} + \bar{\phi}\hat{\rho}^{(1)} \tag{3.4}$$

We have

$$\hat{f}_j^{(1)} = M\left(\sum_{\ell=0}^{4}\chi_\ell\alpha_\ell^{(1)}\bar{\rho}_j + (-1)^{j+1}\frac{1}{2}\bar{\rho}\psi^{(1)}\right)$$

$$\hat{\rho}_i^{(1)} := \int dv\hat{f}_i^{(1)} = \bar{\rho}_i\hat{\rho}^{(1)} + (-1)^{i+1}\frac{1}{2}\bar{\rho}\psi^{(1)}, \quad \hat{u}_i := \int dvv\hat{f}_i^{(1)} = \hat{u}\bar{\rho}_i,$$

$$\hat{T}_i^{(1)} := \int dv\frac{|v|^2 - 3\bar{T}}{3}\hat{f}_i^{(1)} = \bar{\rho}_i\hat{T}^{(1)} \tag{3.5}$$

Equation (2.10) for $n = 2$ gives

$$v\cdot\nabla_r\hat{f}_i^{(1)} - \nabla_r\hat{g}_i^{(1)}\cdot\nabla_v\hat{f}_i^{(0)} = \mathcal{L}\hat{f}_i^{(2)} + J\big(\hat{f}_i^{(1)}, \gamma_{+}^{(1)}\big) + J\big(\gamma_{+}^{(1)}, \hat{f}_i^{(1)}\big) \tag{3.6}$$

Summing on $i = 1, 2$ we get

$$v\cdot\nabla_r\hat{\gamma}_{+}^{(1)} - \nabla_r\hat{g}_{+}^{(1)}\cdot\nabla_v\hat{\gamma}_{+}^{(0)} + \nabla_r\hat{g}_{-}^{(1)}\cdot\nabla_v\hat{\gamma}_{-}^{(0)} = \mathcal{L}\gamma_{+}^{(2)} + J\big(\gamma_{+}^{(1)}, \gamma_{+}^{(1)}\big) \tag{3.7}$$

where $\hat{g}_{\pm}^{(1)} = \hat{U} \int dv \hat{\gamma}_{\pm}^{(1)}$.

In order that Eq. (3.7) be solvable, the compatibility condition

$$\mathcal{P}\big[v \cdot \nabla_r \hat{\gamma}_+^{(1)} - \nabla_r \hat{g}_+^{(1)} \cdot \nabla_v \hat{\gamma}_+^{(0)} + \nabla_r \hat{g}_-^{(1)} \cdot \nabla_v \hat{\gamma}_-^{(0)}\big] = 0$$

has to be fulfilled, where $\mathcal{P}$ is the projector on the null space of $\mathcal{L}$. We have that

$$\mathcal{P}\big[v \cdot \nabla_r \hat{\gamma}_+^{(1)}\big] = M\bar{\rho} \left[\left(1 + \frac{v^2 - 3\bar{T}}{2\bar{T}^2} \right) \nabla_r \cdot \hat{u} + \frac{v}{\bar{T}} \cdot \nabla_r \big(\bar{T} \hat{\rho}^{(1)} + \hat{T}^{(1)} \big) \right],$$

while

$$\mathcal{P}\big[\nabla_r \hat{g}_+^{(1)} \cdot \nabla_v \hat{\gamma}_+^{(0)} - \nabla_r \hat{g}_-^{(1)} \cdot \nabla_v \hat{\gamma}_-^{(0)}\big] = -M\bar{\rho} \frac{v}{\bar{T}} \cdot \nabla_r \hat{g}_+^{(1)} + M\bar{\phi} \frac{v}{\bar{T}} \cdot \nabla_r \hat{g}_-^{(1)}.$$

The compatibility condition implies the condition on the divergence of the velocity field

$$\operatorname{div} \hat{u} = 0 \tag{3.8}$$

and the Boussinesq condition

$$\nabla_r \big(\bar{T} \bar{\rho} \hat{\rho}^{(1)} + \bar{\rho} \hat{T}^{(1)} + \bar{\rho} \hat{g}_+^{(1)} - \bar{\phi} \hat{g}_-^{(1)} \big) = 0 \tag{3.9}$$

We notice that the last condition can be rewritten in terms of the chemical potentials $\mu_i^{(1)}$. The chemical potentials are defined as

$$\mu_i = \frac{\delta \mathcal{F}}{\delta \rho_i} = T(1 + \log \rho_i) + U * \rho_j, \quad i \neq j$$

We have also $\mu_i^\varepsilon := \mu_i(\rho_i^\varepsilon) = \sum_n \varepsilon^n \mu_i^{(n)}$ so that the Boussinesq condition becomes

$$\sum_{i=1}^{2} \bar{\rho}_i \nabla_r \mu_i^{(1)} = -\nabla_r \hat{T}^{(1)} \sum_{i=1}^{2} \bar{\rho}_i \log \bar{\rho}_i$$

We have not yet determined the first order corrections to the hydrodynamic fields. To this end, we consider (2.10) for $n = 3$

$$\partial_t \hat{f}_i^{(1)} + v \cdot \nabla_r \hat{f}_i^{(2)} - \nabla_r \hat{g}_i^{(2)} \cdot \nabla_v \hat{f}_i^{(0)} - \nabla_r \hat{g}_i^{(0)} \cdot \nabla_v \hat{f}_i^{(2)} - \nabla_r \hat{g}^{(1)} \cdot \nabla_v \hat{f}_i^{(1)}$$
$$= J\big(\hat{f}_i^{(1)}, \hat{\gamma}_+^{(2)}\big) + J\big(\hat{\gamma}_+^{(2)}, \hat{f}_i^{(1)}\big) + \mathcal{L} \hat{f}_i^{(3)} \tag{3.10}$$

Since $\hat{f}_i^{(0)}$ is independent of r the term involving the potential is indeed a gradient term and will contribute to the pressure. We replace $\hat{f}_i^{(2)}$ as given by (3.6) in (3.10) and integrate over velocity after multiplying by v. The result is Eq. (1.11).[2]

To identify the pressure we write explicitly the expression of $\gamma_+^{(2)} = \sum_{i=1}^2 \hat{f}_i^{(2)}$

$$\gamma_+^{(2)} = \mathcal{L}^{-1}\big(- 2J\big(\hat{\gamma}_+^{(1)}, \hat{\gamma}_+^{(1)}\big) + P^{\perp}\big(v \cdot \nabla_r \hat{\gamma}_+^{(1)}\big)\big) + \sum_{i=1}^{2} \hat{\ell}_i^{(2)} \tag{3.11}$$

with $\hat{\ell}_i^{(2)}$ in the invariant space and $P^{\perp}$ the projector on the space orthogonal to the invariant space. The gradient of p is given by

$$\begin{aligned}\partial_k p &= \partial_k \frac{|u|^2}{2} + \partial_k \int dv \frac{v^2}{3} \sum_{i=1,2} \hat{\ell}_i^{(2)}(r, v, t) - \sum_{i=1,2} \Bigg[\nabla_r \hat{g}_i^{(2)} \cdot \int dv v_k \nabla_v f_i^{(0)} \\ &\quad - \nabla_r \hat{g}_i^{(0)} \cdot \int dv v_k \nabla_v \hat{f}_i^{(2)} - \nabla_r \hat{g}_i^{(1)} \cdot \int dv v_k \nabla_v \hat{f}_i^{(1)} \Bigg] \qquad (3.12)\\ &= \sum_{i=1,2} \nabla \left[\left\langle \frac{v^2}{3} \hat{\ell}_i^{(2)} \right\rangle + \frac{1}{2}|u|^2 \right] + \sum_i \left[\hat{\rho}_i^{(0)} \nabla_r \hat{g}_i^{(2)} + \hat{\rho}_i^{(2)} \nabla_r \hat{g}_i^{(0)} + \frac{\hat{U}}{2} \nabla_r \big(\hat{\rho}_i^{(1)}\big)^2 \right]\end{aligned}$$

The last term in square brackets is equal to $\nabla_r(\hat{U}\bar{\rho}_i\hat{\rho}_i^{(2)} + \frac{\hat{U}}{2}(\rho_i^{(1)})^2)$. This identifies the pressure up to a constant as

$$p = \sum_{i=1,2} \left[\int dv \frac{v^2}{3} \hat{\ell}_i^{(2)}(r, v, t) + \frac{1}{2}|u|^2 + \hat{U}\bar{\rho}_i \hat{\rho}_i^{(2)} + \frac{\hat{U}}{2} \big(\hat{\rho}_i^{(1)}\big)^2 \right] \tag{3.13}$$

By integrating (3.10) over velocity we get the following equations for $\hat{T}^{(1)}$ and $\hat{\phi}^{(1)}$

$$D_t \phi^{(1)} = D \frac{\bar{\rho}\bar{T} + \hat{U}(\bar{\rho}^2 - \bar{\phi}^2)}{\bar{\rho}^2(\bar{T} + \bar{\rho}\hat{U})} \Delta_r \phi^{(1)} + D \frac{\bar{\phi}}{\bar{\rho}} \frac{1}{\bar{T} + \bar{\rho}\hat{U}} \Delta_r T^{(1)} \tag{3.14}$$

$$\bar{\rho} \left(1 + \frac{1}{\bar{T} + \bar{\rho}\hat{U}} \right) D_t T^{(1)} = k \Delta_r T^{(1)} + \bar{\phi}\hat{U} D_t \phi^{(1)} \tag{3.15}$$

where D is the diffusion coefficient.[2] We have used the Boussinesq condition and

$$\bar{\rho}\partial_t \hat{\rho}^{(1)} + \bar{\rho}\,\mathrm{div}\, \hat{u}^{(2)} = 0 \tag{3.16}$$

where $\hat{u}_i^{(2)} := \int dv v \hat{f}_i^{(2)} = \bar{\rho}\hat{u}^{(2)}$. These equations are similar to the ones in the phase field models,[7] but for the facts that in the (3.14) the non linear term in the concentration is missing and the term proportional to the temperature is replaced by the Laplacian of the temperature times a possibly negative coefficient.

Inner expansion

We now examine the equations in $\mathcal{N}^0(m)$, which will provide the boundary layer corrections $\tilde{f}_i^{(n)}$ and the boundary conditions on the interface for the equations in the bulk as well.

We start from (2.11) for $n = 0$

$$v \cdot \bar{\nu}\partial_z \tilde{f}_i^{(0)} - \partial_z \tilde{g}_i^{(0)}\, \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(0)} = 0 \tag{3.17}$$

where $\bar{\nu} = \nu^{(0)}$. The matching conditions impose $\lim_{z\to\pm\infty} \tilde{f}_i^{(0)}(z, r, t) = (\hat{f}_i^{(0)})^{\pm}(r, t)$. We know that $\hat{f}_i^{(0)}(r, v, t) = \hat{\rho}_i^{(0)} M$, with $\lim_{d^0(r,t)\to 0^{\pm}} \hat{\rho}_i^{(0)} = \bar{\rho}_i^{\pm}$.

Hence, we look for a solution to the Eq. (3.17) on $\mathbb{R} \times \mathbb{R}^3$ (r, t are seen as parameters) with the conditions at infinity

$$\lim_{z\to\pm\infty} \tilde{f}_i^{(0)}(z, r, t) = \bar{\rho}_i^{\pm} M.$$

We first show that there is a unique solution among the class of functions

$$\left\{ S : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^2, \quad \int dv S_i(z, v) = \rho_i(z) \right\}$$

for a given ρ_i. Let $g_i = \tilde{U} * \rho_j$, $j \neq i$. If we consider such two solutions S_i, S_i' in such class and write the equation for the difference $q_i = S_i - S_i'$

$$v \cdot \nu \partial_z q_i - \partial_z g_i \nu \cdot \nabla_v q_i = 0,$$

we obtain for the moments $\langle v_z^n q_i \rangle$ of any order $n \geq 1$

$$\partial_z \langle v_z^{n+1} q_i \rangle = \partial_z g_i n \langle v_z^{n-1} q_i \rangle.$$

Since $\langle q_i \rangle = 0$ we have that $\partial_z \langle v_z^2 q_i \rangle = 0$ and the constant is fixed by the conditions at infinity which are $\lim_{z\to\pm\infty} q_i = 0$. By iteration, this implies $q_i = 0$ and hence the uniqueness.

Then, we observe that the equation with fixed potential g_i has an explicit solution given by $Me^{-g_i/\bar{T}}$, which satisfies the matching conditions and hence is the unique solution (by the previous argument) with density

$$\rho_i = \int dv M e^{-g_i/\bar{T}}.$$

By recalling that $g_i = \tilde{U} * \rho_j$, we realize that ρ_i has to be indeed the front w_i solution of

$$\partial_z w_i + \frac{1}{\bar{T}} w_i \tilde{U} * \partial_z w_j = 0 \tag{3.18}$$

with conditions at infinity $\bar{\rho}_i^{\pm}$. This equation written in the form

$$\partial_z \left(\log w_i + \frac{1}{\bar{T}} \tilde{U} * w_j \right) = 0$$

is exactly Eq. (1.6) in dimension 1. In the following we will often use the notation $\partial_z w_i = w_i'$. In conclusion, we have shown that Eq. (3.17) for the first order correction $\tilde{f}_i^{(0)}$ in the inner expansion with the prescribed boundary condition has the unique solution

$$\tilde{f}_i^{(0)} = M w_i.$$

Next, we study (2.11) for $n = 1$:

$$\bar{V}\partial_z \tilde{f}_i^{(0)} + v \cdot \bar{\nu}\, \partial_z \tilde{f}_i^{(1)} - \partial_z \tilde{g}_i^{(0)} \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(1)} - \partial_z \tilde{g}_i^{(1)}\, \bar{\nu} \cdot \nabla_v \tilde{f}_i^{(0)} = 0 \tag{3.19}$$

The term involving $\nu^{(1)}$: $\nu^{(1)} \cdot (v\partial_z \tilde{f}_i^{(0)} - \partial_z \tilde{g}_i^{(0)} \nabla_v f_i^{(0)})$ is zero because $\tilde{f}_i^{(0)}$ satisfies (3.17). Integrating over v and using the explicit form of $\tilde{f}_i^{(0)}$ we get

$$\partial_z\big(\nu \cdot \tilde{u}_i^{(1)}\big) + \bar{V}\partial_z w_i = 0$$

with $\tilde{u}_i^{(1)} = \int dv v \tilde{f}_i^{(1)}$. Summing on $i = 1, 2$

$$\partial_z\big(\bar{\nu} \cdot \big(\tilde{u}_1^{(1)} + \tilde{u}_2^{(1)}\big)\big) + \bar{V}\partial_z(w_1 + w_2) = 0$$

Integrating over $z \in \mathbb{R}$:

$$\big[\bar{\nu} \cdot \big(\tilde{u}_1^{(1)} + \tilde{u}_2^{(1)}\big)\big]_{-\infty}^{+\infty} = -\bar{V}[w_1 + w_2]_{-\infty}^{+\infty}.$$

By matching conditions, $\tilde{u}_i^{(1)} \to (\hat{u}\bar{\rho}_i)^{\pm}$ and $w_1 + w_2 \to \bar{\rho}$, hence we have

$$\bar{\rho}[\bar{\nu} \cdot \hat{u}^+ - \bar{\nu} \cdot \hat{u}^-] = -\bar{V}\big[w_1 + w_2\big]_{-\infty}^{+\infty} = 0, \tag{3.20}$$

Then we have

$$\bar{\rho}[\bar{\nu} \cdot \bar{u}]_-^+ = 0 \quad r \in \bar{\Gamma}_t$$

The symbol $[h]_-^+$ stands for the jump of a function h across $\bar{\Gamma}_t$. Hence, $\hat{u}$ is continuous on $\bar{\Gamma}_t$. Moreover, it is also true that

$$\big[\bar{\nu} \cdot \hat{u}\bar{\rho}_i\big]_-^+ = -\bar{V}[w_i]_{-\infty}^{+\infty}$$

which implies due to the continuity of $\bar{\nu} \cdot \hat{u}$

$$\bar{\nu} \cdot \hat{u}(r, t) = -\bar{V}(r, t), \quad r \in \bar{\Gamma}_t \tag{3.21}$$

We remark that $\int_{\bar{\Gamma}_t} ds \bar{V} = 0$ because $\hat{u}$ is divergence free. This implies that the area enclosed by $\bar{\Gamma}_t$ is conserved by the limiting dynamics.

We look for a solution to (3.19) matching $\hat{f}_i^{(1)}$ and hence we try with a combination of collision invariants, namely a solution of the form

$$\tilde{f}_i^{(1)} = M\left(\tilde{\rho}_i^{(1)} + \frac{v \cdot \tilde{u}_i^{(1)}}{\bar{T}} + \frac{|v|^2 - 3\bar{T}}{2\bar{T}^2}\tilde{T}_i^{(1)}\right)$$

Plugging in (3.19) the previous expression we get an expansion of the l.h.s. of (3.19) with respect the basis ($\ell \neq z$)

$$M,\ Mv_z,\ Mv_z v_\ell,\ Mv_z^2,\ Mv_z|v|^2$$

in the form

$$
\begin{aligned}
M\Bigg[\, & \bar V \partial_z \tilde\rho_i^{(0)} - \frac{1}{\bar T}\big(\tilde u_i^{(1)}\big)_z \partial_z \tilde U * \tilde\rho_j^{(0)} \\
& + v_z\left(\partial_z \tilde D_i + \frac{1}{\bar T}\tilde D_i \partial_z \tilde U * \tilde\rho_j^{(0)} - \frac{1}{\bar T^2}\tilde T_i^{(1)}\partial_z \tilde U * \tilde\rho_j^{(0)} + \frac{1}{\bar T}\tilde\rho_i^{(0)}\partial_z \tilde U * \tilde\rho_j^{(1)}\right) \\
& + \sum_\ell v_z v_\ell \left(\partial_z\left(\frac{1}{\bar T}\tilde u_i^{(1)}\right)_\ell + \frac{1}{\bar T^2}\big(\tilde u_i^{(1)}\big)_\ell \partial_z \tilde U * \tilde\rho_j^{(0)}\right) \\
& + v_z |v|^2 \frac{1}{2\bar T^2}\left(\partial_z \tilde T_i^{(1)} + \frac{1}{\bar T}\tilde T_i^{(1)}\partial_z \tilde U * \tilde\rho_j^{(0)}\right)\Bigg]
\end{aligned}
\tag{3.22}
$$

where $\tilde D_i := \tilde\rho_i^{(1)} - \frac{3}{2\bar T}\tilde T_i^{(1)}$. Equating to zero the coefficients gives a set of equations for $\tilde\rho_i^{(1)}$, $\tilde u_i^{(1)}$, $\tilde T_i^{(1)}$, the inner counterpart of the quantities defined in (3.5). We have two conditions involving $\bar\nu \cdot \tilde u_i^{(1)}$:

$$w_i' \bar V - \frac{1}{\bar T}\tilde U * w_j' \nu \cdot \tilde u_i^{(1)} = 0 \tag{3.23}$$

$$\partial_z \tilde u_i^{(1)} + \frac{1}{\bar T} U * w_j' \tilde u_i^{(1)} = 0 \tag{3.24}$$

to be solved under the matching conditions $\lim_{z\to\pm\infty} \tilde u_i^{(1)} = (\hat u)^{\pm}\bar\rho_i$. The second Eq. (3.24) can be rewritten using the equation for the front (3.18) as

$$w_i \partial_z \frac{\tilde u_i^{(1)}}{w_i} = 0$$

For any vector $\bar\tau$ tangent to $\bar\Gamma_t$ we have

$$\frac{\bar\tau \cdot \tilde u_i^{(1)}}{w_i} = \text{const}$$

By matching conditions, $\lim_{z\to\pm\infty} \bar\tau \cdot \tilde u_i^{(1)} = (\bar\tau \cdot \hat u \bar\rho_i)^{\pm}$ and, by the asymptotic behavior of w_i we get that the constant has to coincide with $(\bar\tau \cdot \hat u)^{\pm}$ and, as a consequence, $(\bar\tau \cdot \hat u)^{+} = (\bar\tau \cdot \hat u)^{-}$. Hence, also the tangential components of $\hat u$ are continuous across the interface $\bar\Gamma_t$ and

$$\bar\tau \cdot \tilde u_i^{(1)}(z, r, t) = (\bar\tau \cdot \hat u)\big|_{\bar\Gamma_t} w_i.$$

The normal projection of the second Eq. (3.24) gives that $\frac{\bar\nu \cdot \tilde u_i^{(1)}}{w_i}$ is a constant w.r.t. z and the first equation fixes the value of the constant. In fact, by using (3.18) we write (3.23) as

$$w_i'\left(\bar V + \frac{\nu \cdot \tilde u_i^{(1)}}{w_i}\right) = 0$$

showing that the constant is $\dfrac{\bar{\nu}\cdot\tilde{u}_i^{(1)}}{w_i} = -\bar{V}$. Then, the matching condition forces $-\bar{V} = \bar{\nu}\cdot\hat{u}, r \in \bar{\Gamma}_t$ which is Eq. (3.21). In conclusion,

$$\bar{\nu}\cdot\tilde{u}_i^{(1)} = \nu\cdot\hat{u}\big|_{\Gamma_t} w_i, \quad r \in \mathcal{N}^0(m).$$

Now we examine the condition for $\tilde{T}_i^{(1)}$:

$$\partial_z \tilde{T}_i^{(1)} + \frac{1}{\bar{T}}\tilde{T}_i^{(1)}\partial_z \tilde{U} * \tilde{\rho}_j^{(0)} = 0$$

that we can write again as

$$w_i\,\partial_z \frac{\tilde{T}_i^{(1)}}{w_i} = 0$$

so that $\dfrac{\tilde{T}_i^{(1)}}{w_i}$ has to be constant and the constant is determined as before through the matching conditions $\tilde{T}_i^{(1)} \to (\bar{\rho}_i \hat{T}^{(1)})^{\pm}$. In conclusion, $\hat{T}^{(1)}$ is continuous on $\bar{\Gamma}_t$ and

$$\tilde{T}_i^{(1)} = \hat{T}^{(1)}\big|_{\bar{\Gamma}_t} w_i \quad r \in \mathcal{N}^0(m).$$

The condition for $\tilde{\rho}_i^{(1)}$ is

$$\partial_z \tilde{\rho}_i^{(1)} + \frac{1}{\bar{T}}\tilde{U} * w_j' \tilde{\rho}_i^{(1)} - \frac{3}{2\bar{T}}\partial_z \tilde{T}_i^{(1)} - \frac{3}{2\bar{T}^2}\tilde{U} * w_j' \tilde{T}_i^{(1)} - \frac{1}{\bar{T}^2}\tilde{U} * w_j' \tilde{T}_i^{(1)} + \frac{1}{\bar{T}}\tilde{U} * \partial_z \tilde{\rho}_j^{(1)} w_i = 0 \tag{3.25}$$

The previous equation can be written as

$$w_i\,\partial_z \left[\frac{\tilde{\rho}_i^{(1)}}{w_i} + \frac{1}{T}\tilde{U} * \tilde{\rho}_j^{(1)}\right] - \frac{3}{2\bar{T}}\partial_z \tilde{T}_i^{(1)} - \frac{5}{2\bar{T}^2}\tilde{T}_i^{(1)}\tilde{U} * w_j' = 0$$

or, by using the equation for $\tilde{T}_i^{(1)}$,

$$w_i\,\partial_z \tilde{h}_i^{(1)} + \partial_z \tilde{T}_i^{(1)} = 0 \tag{3.26}$$

where $\tilde{h}_i^{(1)} := \bar{T}\dfrac{\tilde{\rho}_i^{(1)}}{w_i} + U * \tilde{\rho}_j^{(1)}$. Since $\tilde{T}_i^{(1)}$ is known the previous relation allows to find $\tilde{h}_i^{(1)}$. Then, $\tilde{\rho}_i^{(1)}$ will be obtained by solving

$$\left(\mathcal{Q}\tilde{\rho}^{(1)}\right)_i = \tilde{h}_i^1 \tag{3.27}$$

where the matrix valued operator $\mathcal{Q}$ is defined on the couples of functions $q = (q_1, q_2)$ as

$$(\mathcal{Q}q)_i = \bar{T} q_i (w_i)^{-1} + \tilde{U} * q_j.$$

The operator $\mathcal{Q}$ is symmetric with respect to the component-wise inner product in $L_2(\mathbb{R})$ and has a zero mode since $w' = (w'_1, w'_2)$ is such that $\mathcal{Q}w' = 0$. Therefore the equation $\mathcal{Q}\tilde{\rho}^{(1)} = q$ has a solution only if the compatibility condition

$$\sum_{i=1,2} \int \mathrm{d}z \tilde{h}_i^{(1)} w'_i(z) = 0. \tag{3.28}$$

is satisfied. Summing on $i = 1, 2$, integrating over z in $\mathbb{R}$ (3.26) and using (3.28), we obtain

$$\sum_{i=1,2} \left[w_i \tilde{h}_i^{(1)} \right]_{-\infty}^{+\infty} = - \sum_{i=1,2} \left[\tilde{T}_i^{(1)} \right]_{-\infty}^{+\infty} = 0 \tag{3.29}$$

We now introduce $\hat{h}_i^{(1)} = \bar{T} \dfrac{\hat{\rho}_i^{(1)}}{\bar{\rho}_i} + \hat{U} \hat{\rho}_j^{(1)}$. By matching conditions

$$\lim_{z \to \pm\infty} \tilde{h}_i^{(1)}(z, r, t) = \left(\hat{h}_i^{(1)}\right)^{\pm} = \left(\bar{T} \frac{\hat{\rho}_i^{(1)}}{\bar{\rho}_i} + \hat{U} \hat{\rho}_j^{(1)} \right)^{\pm} \tag{3.30}$$

The quantity

$$\sum_{i=1,2} \bar{\rho}_i \hat{h}_i^{(1)} + \bar{\rho} \hat{T}^{(1)} = \bar{T} \bar{\rho} \hat{\rho}^{(1)} + \bar{\rho} \hat{g}_+^{(1)} - \bar{\phi} \hat{g}_-^{(1)} + \bar{\rho} \hat{T}^{(1)}$$

is exactly the first correction to the effective pressure $\hat{p}^{(1)}$ appearing in the Boussinesq condition (3.9). The condition (3.30) and the continuity of $\hat{T}^{(1)}$ imply the continuity of $\hat{p}^{(1)}$ on $\bar{\Gamma}_t$ and together with (3.9) say that $\hat{p}^{(1)}$ has a constant value in Ω.

We compute now the value of the difference $\hat{h}_1^{(1)} - \hat{h}_2^{(1)}$ on $\bar{\Gamma}_t$. We start from the compatibility condition (3.28) and add and subtract the term $\sum_{i=1}^{2} \int dv w'_i \bar{\rho}_i \hat{T}^{(1)}|_{\bar{\Gamma}_t} \log w_i$

$$\sum_{i=1}^{2} \int dz w'_i \left(\tilde{h}_i^{(1)} + \bar{\rho}_i \hat{T}^{(1)}|_{\bar{\Gamma}_t} \log w_i \right) - \sum_{i=1}^{2} \bar{\rho}_i \hat{T}^{(1)}|_{\bar{\Gamma}_t} \int dz w'_i \log w_i = 0 \tag{3.31}$$

By using the explicit form of $\tilde{T}_i^{(1)} = w_i \hat{T}^{(1)}|_{\bar{\Gamma}_t}$ the Eq. (3.26) for $\tilde{h}_i^{(1)}$, divided by w_i, becomes

$$\partial_z \tilde{h}_i^{(1)} + \frac{1}{w_i} w'_i \hat{T}^{(1)}|_{\bar{\Gamma}_t} = \partial_z \left(\tilde{h}_i^{(1)} + \hat{T}^{(1)}|_{\bar{\Gamma}_t} \log w_i \right) = 0 \tag{3.32}$$

Since $\tilde{h}_i^{(1)} + \hat{T}^{(1)}\big|_{\bar{\Gamma}_t} \log w_i$ is a constant, that we call α_i, we can write (3.31) as

$$0 = \sum_{i=1}^{2} \alpha_i [w_i]_{-\infty}^{+\infty} - \sum_{i=1}^{2} \hat{T}^{(1)}\big|_{\bar{\Gamma}_t}([w_i \log w_i]_{-\infty}^{+\infty} - [w_i]_{-\infty}^{+\infty})$$

$$= \sum_{i=1}^{2} \alpha_i [w_i]_{-\infty}^{+\infty} = (\alpha_1 - \alpha_2)\big(\rho_1^+ - \rho_1^-\big),$$

because $\rho_1^\pm = \rho_2^\mp$. This implies $\alpha_1 = \alpha_2$ and

$$\tilde{h}_1^{(1)} - \tilde{h}_2^{(1)} = \hat{T}^{(1)}\big|_{\bar{\Gamma}_t} \log \frac{w_2}{w_1}.$$

By matching conditions

$$\lim_{z\to\pm\infty} \tilde{h}_i^{(1)}(z, r, t) = \big(\hat{h}_i^{(1)}\big)^\pm, \quad \lim_{z\to\pm\infty} \frac{w_2}{w_1} = \frac{(\bar{\rho}_2)^\pm}{(\bar{\rho}_1)^\pm}.$$

Since $\hat{\phi}^{(1)} = \hat{h}_1^{(1)} - \hat{h}_2^{(1)}$ we have that

$$\big[\hat{\phi}^{(1)}\big]_-^+ = \hat{T}^{(1)}\big|_{\bar{\Gamma}_t} \left[\log \frac{\bar{\rho}_2}{\bar{\rho}_1}\right]_-^+ \neq 0$$

on $\bar{\Gamma}_t$ which means that $\bar{\phi}^{(1)}$ jumps from a value $|\bar{\phi}^{(1)}|$ to $-|\bar{\phi}^{(1)}|$. On the other hand $\bar{\rho}_1\alpha_1 + \bar{\rho}_2\alpha_2 = \hat{p}^{(1)}|_{\bar{\Gamma}_t}$ and this fixes the value of $\alpha_1 + \alpha_2$ and hence of α_i as $\dfrac{1}{2\bar{\rho}}\hat{p}^{(1)}|_{\bar{\Gamma}_t}$. We observe that $\bar{\rho}^{(1)}$ is continuous on $\bar{\Gamma}_t$.

This analysis provides the boundary conditions for the set of coupled Eqs. (3.9), (3.14) and (3.15). Once solved, we will have through the matching conditions the conditions at infinity to find the solution of (3.26). We see from (3.32) that $\tilde{h}_i^{(1)}$ is a function decaying exponentially to its limit value.

We need now to show how to solve the Eq. (3.27). Let us denote

$$\tilde{h}_i^{(1),\pm\infty} = \lim_{z\to\pm\infty} \tilde{h}_i^{(1)}(z), \quad \tilde{\rho}_i^{(1),\pm\infty} = \lim_{z\to\pm\infty} \tilde{\rho}_i^{(1)}(z)$$

Clearly, by passing to the limit in (3.27) and using $\int dz \tilde{U}(z) = 1$, we see that $\tilde{\rho}_i^{(1),\pm\infty}$ satisfy:

$$\bar{T}\bar{\rho}_i^{-1}\tilde{\rho}_i^{(1),\pm\infty} + \tilde{\rho}_j^{(1),\pm\infty} = \tilde{h}_i^{(1),\pm\infty}.$$

We introduce the discontinuous functions $R_i^{(1)}(z)$ and $H_i^{(1)}(z)$ as follows:

$$R_i^{(1)}(z) = \begin{cases} \tilde{\rho}_i^{(1),+\infty} & \text{if } z > 0, \\ \tilde{\rho}_i^{(1),-\infty} & \text{if } z < 0 \end{cases}, \qquad H_i^{(1)}(z) = \begin{cases} \tilde{h}_i^{(1),+\infty} & \text{if } z > 0, \\ \tilde{h}_i^{(1),-\infty} & \text{if } z < 0 \end{cases}$$

and

$$\theta_i^{(1)} = \tilde{\rho}_i^{(1)} - R_i^{(1)}, \quad \lambda_i^{(1)} = \tilde{h}_i^{(1)} - H_i^{(1)}.$$

The functions $\theta_i^{(1)}$ have to solve

$$\mathcal{Q}\theta^{(1)} = \lambda^{(1)} + \mathcal{U}R^{(1)},$$

with $(\mathcal{U}R)_i = -\tilde{U} * R_j + R_i(\bar{\rho}_i^{-1} - w_i^{-1})$. Since $\tilde{h}^{(1)}$ satisfies the compatibility condition (3.28), by the Fredholm alternative $\theta^{(1)}$ can be found uniquely in $L_2(\mathbb{R})$ up to a term in the null space of $\mathcal{Q}$. Let us denote by $\mathcal{Q}^{-1}\tilde{h}^{(1)}$ the so found solution to (3.27) orthogonal to w'. Then $\tilde{\rho}_i^{(1)}$ are given in terms of $\tilde{h}_i^{(1)}$ as

$$\tilde{\rho}_i^{(1)} = \mathcal{Q}^{-1}\tilde{h}_i^{(1)} + \gamma w_i'$$

with γ independent of z to be determined. It can be fixed by imposing in $z = 0$ $(\tilde{\rho}_1^{(1)} - \tilde{\rho}_2^{(1)}) = 0$. We are allowed to do that because we choose $\rho_1^\varepsilon = \rho_2^\varepsilon$. Finally, since $\lambda^{(1)}$ and w' decay exponentially and $\mathcal{U}R$ is the sum of the compactly supported $\tilde{U} * R$ and an exponentially decaying term, one can show that $\theta^{(1)}$ also decays exponentially and hence $\tilde{\rho}_i^{(1)}$ decays exponentially to its limiting value.

The first order in the inner expansion is not yet completely determined because to find $\tilde{u}_i^{(1)}$ we need to know the value of the velocity field $\hat{u}$ on $\bar{\Gamma}_t$. This will be provided by the solution of (1.11), once we will have all the boundary conditions we need. We still miss the ones for the pressure p. For that, we consider the next order equation in the inner expansion, namely Eq. (2.11) for $n = 2$

$$\begin{aligned} &\bar{V}\partial_z \tilde{f}_i^{(1)} + V^{(1)}\partial_z \tilde{f}_i^{(0)} + \bar{\nu}\cdot v\partial_z \tilde{f}_i^{(2)} - \partial_z \tilde{g}_i^{(0)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(2)} - \partial_z \tilde{g}_i^{(2)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(0)} \\ &\quad -\partial_z \tilde{g}_i^{(1)}\bar{\nu}\cdot\nabla_v \tilde{f}_i^{(1)} - \nu^{(1)}\cdot\left[v\partial_z \tilde{f}_i^{(1)} + \nabla_v \tilde{f}_i^{(1)}\partial_z \tilde{g}_i^{(0)} + \nabla_v \tilde{f}_i^{(0)}\partial_z \tilde{g}_i^{(1)}\right] \\ &= J\big(\tilde{f}_i^{(0)}, \tilde{\gamma}_+^{(1)}\big) + J\big(\tilde{f}_i^{(1)}, \tilde{\gamma}_+^{(0)}\big) \end{aligned} \tag{3.33}$$

Again, the terms involving $\nu^{(2)}$ disappear. From (3.33) multiplying by $v\cdot\bar{\nu}$ and integrating over v we have, because of the orthogonality of $\nabla_r d^{(0)}$ and $\nabla_r d^{(1)}$ and the particular form of $f_i^{(1)}$,

$$\bar{V}\partial_z\big(\bar{\nu}\cdot\tilde{u}_i^{(1)}\big) + \partial_z \int dv(\bar{\nu}\cdot v)^2 \tilde{f}_i^{(2)} + \tilde{\rho}_i^{(2)}\partial_z \tilde{g}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z \tilde{g}_i^{(2)} + \rho_i^{(1)}\partial_z \tilde{g}_i^{(1)} = 0. \tag{3.34}$$

We decompose $\tilde{f}_i^{(2)}$ in a part orthogonal to the invariant space $\tilde{t}_i^{(2)}$ and in a part on the invariant space, write the latter as

$$M(v)\Big(\tilde{\rho}_i^{(2)} + \tilde{u}_i^{(2)}\cdot\frac{v}{\bar{T}} + \tilde{T}_i^{(2)}\frac{v^2 - 3\bar{T}}{2\bar{T}^2}\Big)$$

and introduce the function

$$\tilde{h}_i^{(2)} = \bar{T}\frac{\tilde{\rho}_i^{(2)}}{\tilde{\rho}_i^{(0)}} + \tilde{g}_i^{(2)}.$$

We recall the matching condition for $\tilde{f}_i^{(2)}$

$$\tilde{f}_i^{(2)} \to \big(\hat{f}_i^{(2)}\big)^{\pm} + \bar{\nu}\cdot\big(\nabla_r \hat{f}_i^{(1)}\big)^{\pm} d^{(1)}.$$

Since the same relation holds for $\tilde{h}_i^{(2)}$ and $\tilde{T}_i^{(2)}$ we have that

$$\sum_{i=1}^{2} w_i \tilde{h}_i^{(2)} \to \sum_{i=1}^{2}\big(\bar{\rho}_i \hat{h}_i^{(2)} + \bar{\rho}_i \hat{T}^{(2)}\big)^{\pm} + \sum_{i=1}^{2} d^{(1)}\bar{\nu}\cdot\nabla_r\big(\bar{\rho}_i \hat{h}_i^{(1)} + \bar{\rho}_i \hat{T}^{(1)}\big)^{\pm}.$$

We recognize in the second term on the r.h.s. the derivative of $\hat{p}^{(1)}$ which is zero by (3.9) while the expression in the parenthesis in the first term is exactly the contribution to p of the invariant part of $\hat{f}_i^{(2)}$.

Equation (3.34) can be written in terms of $\tilde{h}_i^{(2)}$ as

$$\bar{V}\partial_z\big(\bar{\nu}\cdot\tilde{u}_i^{(1)}\big) + w_i\partial_z\tilde{h}_i^{(2)} + \partial_z\tilde{T}_i^{(2)} + \tilde{\rho}_i^{(1)}\partial_z\tilde{g}_i^{(1)} + \partial_z\int dv(v\cdot\bar{\nu})^2\tilde{t}_i^{(2)} = 0 \qquad (3.35)$$

The integral over z gives

$$\bar{V}\big[\bar{\nu}\cdot\tilde{u}_i^{(1)}\big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dz\,\tilde{\rho}_i^{(1)}\partial_z\tilde{g}_i^{(1)} + \big[w_i\tilde{h}_i^{(2)} + \tilde{T}_i^{(2)}\big]_{-\infty}^{+\infty}$$
$$- \int_{-\infty}^{+\infty} dz\,w_i'\tilde{h}_i^{(2)} + \left[\int dv(v\cdot\bar{\nu})^2\tilde{t}_i^{(2)}\right]_{-}^{+}$$

After summing on i we have that the first term vanishes because of the matching condition and the continuity of $\hat{u}\bar{\rho}$. The second term can be written by the properties of the convolution as

$$-\frac{1}{2}\int_{-\infty}^{+\infty} dz\,\partial_z\big[\tilde{\rho}_i^{(1)}U * \tilde{\rho}_j^{(1)}\big] = -\frac{1}{2}\sum_{i=1}^{2}\big[\tilde{\rho}_i^{(1)}\tilde{U} * \tilde{\rho}_j^{(1)}\big]_{-\infty}^{+\infty}$$

and is equal by matching conditions to

$$\hat{U}\big[\hat{\rho}_1^{(1)}\hat{\rho}_2^{(1)}\big]_{-}^{+} = 0$$

We have used that $\hat{\rho}_1^{(1)}\hat{\rho}_2^{(1)} = (\bar{\rho}\hat{\rho}^{(1)})^2 - (\hat{\phi}^{(1)})^2$ is continuous. We are left with

$$\sum_{i=1}^{2}\left[\bar{\rho}_i\hat{h}_i^{(2)} + \bar{\rho}_i\hat{T}^{(2)} + \int dv(v\cdot\bar{\nu})^2\hat{t}_i^{(2)}\right]_{-}^{+} - \sum_{i=1}^{2}\int_{-\infty}^{+\infty} dz\,w_i'\tilde{h}_i^{(2)} = 0 \qquad (3.36)$$

where $\hat{t}_i^{(2)}$ is the projection of $\hat{f}_i^{(2)}$ on the orthogonal to the invariant space. To deal with the last term we start from the relation

$$(\mathcal{Q}\tilde{\rho})_i^{(2)} = \tilde{h}_i^{(2)} - \bar{K}\int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)w_j(z')$$

where we used the definition of $\tilde{h}_i^{(2)}$ and the form of $\tilde{g}_i^{(2)}$ given in Appendix A. Since $\mathcal{Q}w' = 0$ we have the compatibility condition

$$\sum_{i=1}^{2}\int dz\tilde{h}_i^{(2)}w_i' = \bar{K}\sum_{i=1}^{2}\int dzdz'w_i'(z)(z'-z)\tilde{U}(|z'-z|)w_j(z')$$

The last term is equal to $\bar{K}\sigma$ (see Appendix B). Replacing in (3.36) we have that

$$\sum_{i=1}^{2}\left[\bar{\rho}_i\hat{h}_i^{(2)} + \bar{\rho}_i\hat{T}^{(2)} + \int dv(v\cdot\bar{\nu})^2\hat{t}_i^{(2)}\right]_-^+ = \bar{K}\sigma$$

It is a matter of a computation based on the expression (3.11) to show that

$$\sum_{i=1}^{2}\left[\int dv(v\cdot\bar{\nu})^2\hat{t}_i^{(2)}\right]_-^+ = \left[\frac{1}{2}|\hat{u}|^2 - \eta\nu\cdot\nabla_r(\hat{u}\cdot\bar{\nu})\right]_-^+. \tag{3.37}$$

The first term on the r.h.s. contributes to the pressure so that (see (3.13))

$$\left[\sum_{i=1}^{2}\left(\bar{\rho}_i\hat{h}_i^{(2)} + \bar{\rho}_i\hat{T}^{(2)}\right) + \frac{1}{2}|\hat{u}|^2\right]_-^+ = [p]_-^+ .$$

The second term instead is zero, due to the continuity of the derivatives of $\hat{u}$, which follows from the continuity of $\hat{u}$ and the condition div $\hat{u} = 0$. Putting all the terms together we get the Laplace's law (1.13).

At this point, we have obtained the last equation of the free boundary value problem (1.11), (1.12) and (1.13) and also determined completely the first order terms of the expansion $\tilde{f}_i^{(1)}$ and $\hat{f}_i^{(1)}$.

4. HIGHER ORDER TERMS

Following an analogous procedure we can construct the higher order terms. Let us discuss briefly the second order term. We start by examining the equations in the bulk at second order. The equations for $\hat{u}^{(2)}$, $\hat{T}^{(2)}$, $\hat{\rho}_i^{(2)}$ will be linearized equations with boundary conditions on $\bar{\Gamma}_t$. Their derivation is standard[(2)] and to lighten the formulas we will discuss explicitly only the simpler case in which we choose at initial time $\hat{\rho}^{(1)} = 0$ and $\hat{T}^{(1)} = 0$, so that they will stay zero at any time because Eqs. (3.14), (3.15) and (3.9) admit the identically zero solution. To get the equation for the velocity field $\hat{u}^{(2)}$ we multiply (2.10), $n = 4$, by v, sum over the index i and integrate over velocity

$$\partial_t \hat{u}^{(2)} + \nabla_r \int dv\, v \otimes v \hat{\gamma}_+^{(3)} - \sum_{i=1}^{2} \hat{F}_i^{(3)} \bar{\rho}_i = 0. \tag{4.1}$$

In above equation some of the force terms drop because $\rho_i^{(1)} = 0$ and $\rho_i^{(0)}$ are constant. The expression of $\hat{\gamma}_+^{(3)} = \sum_{i=1}^{2} \hat{f}_i^{(3)}$ is found from Eq. (2.10), $n = 3$, after summing over i:

$$\hat{\gamma}_+^{(3)} = \mathcal{L}^{-1}\Big(-Q\big(\hat{\gamma}_+^{(2)}, \hat{\gamma}_+^{(1)}\big) + P^{\perp}\big(v \cdot \nabla_r \hat{\gamma}_+^{(2)}\big)\Big) + \sum_{i=1}^{2} \hat{\ell}_i^{(3)}, \tag{4.2}$$

with $Q(f, g) = J(f, g) + J(g, f)$, $P^{\perp}$ the projector on the orthogonal to the invariant space, $\hat{\ell}_i^{(3)}$ belongs to the invariant space and has the form

$$\hat{\ell}_i^{(3)} = M(v)\Big(\hat{\rho}_i^{(3)} + \hat{u}_i^{(3)} \cdot \frac{v}{\bar{T}} + \hat{T}_i^{(3)} \frac{v^2 - 3\bar{T}}{2\bar{T}^2}\Big)$$

We separate the tensor $\int dv v \otimes v \hat{\gamma}_+^{(3)}$ into its traceless and diagonal parts:

$$\int dv\, v \otimes v \hat{\gamma}_+^{(3)} = \int dv \big(v \otimes v - \frac{1}{3}|v|^2\big)\hat{\gamma}_+^{(3)} + \int dv \frac{1}{3}|v|^2 \hat{\gamma}_+^{(3)}.$$

We put $A(v) = v \otimes v - \frac{1}{3}|v|^2$. Since $A(v)$ is orthogonal to the invariant space, we have

$$\int dv A(v) \hat{\gamma}_+^{(3)} = -\int dv A(v) \mathcal{L}^{-1} Q\big(\hat{\gamma}_+^{(2)}, \hat{\gamma}_+^{(1)}\big) + \int dv A(v) \mathcal{L}^{-1} P^{\perp}\big(v \cdot \nabla_r \hat{\gamma}_+^{(2)}\big) \tag{4.3}$$

By replacing (4.2) in (4.4) we get, after some computation,

$$\partial_t \hat{u}^{(2)} + u \cdot \nabla_r \hat{u}^{(2)} + \nabla_r \hat{P}^{(3)} = \eta \Delta_r \hat{u}^{(2)} + N \tag{4.4}$$

The first term in (4.3) gives the transport term, a contribution to the pressure, that we denote by ω, and a source term N depending only on quantities at lower orders. The second term is responsible for the dissipative term in (4.4). The diagonal part of the tensor $\int dv v \otimes v \hat{\gamma}_+^{(3)}$ gives another contribution to the pressure. Therefore, the expression of the pressure $\hat{p}^{(3)}$ is

$$\hat{p}^{(3)} = \omega + \sum_{i=1}^{2} \big(\hat{U} \bar{\rho}_i \hat{\rho}_i^{(3)}\big) + \int dv \frac{v^2}{3} \sum_{i=1,2} \hat{\ell}_i^{(3)} \tag{4.5}$$

We can write the pressure in terms of $\hat{h}_i^{(3)} = \bar{T}\dfrac{\hat{\rho}_i^{(3)}}{\bar{\rho}_i} + \hat{g}_i^{(3)}$ as

$$\hat{p}^{(3)} = \sum_{i=1}^{2} \left(\bar{\rho}_i \hat{h}_i^{(3)} + \bar{\rho}\hat{T}_i^{(3)}\right) + \omega \tag{4.6}$$

The condition (3.16) gives in this special case div $\hat{u}^{(2)} = 0$. Therefore, $\hat{p}^{(3)}$ has again the role of a Lagrangian multiplier, so it is one of the unknown to be found after prescribing the appropriate boundary conditions. They will be provided by Eq. (3.33) in the inner expansion.

By integrating (3.33) over v and remembering that in this section $\tilde{f}_i^{(1)} = \dfrac{M}{\bar{T}} v \cdot \tilde{u}^{(1)}$, we get

$$V^{(1)}\partial_z w_i - \nu^{(1)} \cdot \partial_z \tilde{u}_i^{(1)} + \partial_z\left(\bar{\nu} \cdot \tilde{u}_i^{(2)}\right) = 0, \tag{4.7}$$

with $\tilde{u}_i^{(2)} = \int dv v \tilde{f}_i^{(2)}$. This equation will fix the correction to the velocity $V^{(1)}$. In fact, by integrating over z and taking the difference in (4.7) we get

$$2V^{(1)}|\bar{\phi}| + [\bar{\nu} \cdot \hat{u}^{(2)}\bar{\phi}]_-^+ - 2|\bar{\phi}|\nu^{(1)} \cdot \hat{u} + d^{(1)}[\bar{\phi}\bar{\nu} \cdot \nabla_r \hat{u}]_-^+ = 0, \quad r \in \bar{\Gamma}_t$$

Moreover, summing on i and using the matching conditions (2.9) gives

$$\sum_{i=1}^{2} \left[\bar{\nu} \cdot \hat{u}^{(2)}\bar{\rho}_i\right]_-^+ + d^{(1)}\left[\bar{\nu} \cdot \sum_{i=1}^{2} \nabla_r(\bar{\nu} \cdot \hat{u})\bar{\rho}_i\right]_-^+ = 0. \tag{4.8}$$

This relation gives the continuity of $\bar{\nu} \cdot \hat{u}^{(2)}$ because the second term is zero. By using the continuity of $\bar{\nu} \cdot \hat{u}^{(2)}$, $\hat{u}$ and $\bar{\nu} \cdot \nabla_r \hat{u}$ we can write velocity $V^{(1)}$ as

$$V^{(1)} = -\bar{\nu} \cdot \hat{u}^{(2)} + \nu^{(1)} \cdot \hat{u} - d^{(1)}\bar{\nu} \cdot \nabla_r \hat{u}.$$

Notice that in the previous relation both $\bar{\nu} \cdot \hat{u}^{(2)}$ and $\nu^{(1)}$ have yet to be determined.

To get the boundary condition for $\hat{p}^{(3)}$, we look at Eq. (2.11) for $n = 3$. Multiplying by $v \cdot \bar{\nu}$ and integrating over v we have

$$\bar{V}\partial_z\left(\bar{\nu} \cdot \tilde{u}_i^{(2)}\right) + V^{(1)}\partial_z\left(\bar{\nu} \cdot \tilde{u}_i^{(1)}\right) + \partial_z \int dv(\bar{\nu} \cdot v)^2 \tilde{f}_i^{(3)} + \tilde{\rho}_i^{(3)}\partial_z \tilde{g}_i^{(0)} + \tilde{\rho}_i^{(0)}\partial_z \tilde{g}_i^{(3)} = 0. \tag{4.9}$$

As before, we decompose $\tilde{f}_i^{(3)}$ in a part on the invariant space

$$M(v)\left(\tilde{\rho}_i^{(3)} + \tilde{u}_i^{(3)} \cdot \frac{v}{\bar{T}} + \tilde{T}_i^{(3)}\frac{v^2 - 3\bar{T}}{2\hat{T}^2}\right)$$

and in a part $\tilde{t}_i^{(3)}$ orthogonal to the invariant space and introduce the function

$$\tilde{h}_i^{(3)} = \bar{T}\frac{\tilde{\rho}_i^{(3)}}{\tilde{\rho}_i^{(0)}} + \tilde{g}_i^{(3)}.$$

We write (4.9) in terms of $\tilde{h}_i^{(3)}$ as

$$\bar{V}\partial_z\big(\bar{\nu}\cdot\tilde{u}_i^{(2)}\big) + V^{(1)}\partial_z\big(\bar{\nu}\cdot\tilde{u}_i^{(1)}\big) + w_i\partial_z\tilde{h}_i^{(3)} + \partial_z\tilde{T}_i^{(3)} + \partial_z\int dv(\bar{\nu}\cdot v)^2\tilde{t}_i^{(3)} = 0 \tag{4.10}$$

The integral over z gives

$$\bar{V}\big[\bar{\nu}\cdot\tilde{u}_i^{(2)}\big]_{-\infty}^{+\infty} + V^{(1)}\big[\bar{\nu}\cdot\tilde{u}_i^{(1)}\big]_{-\infty}^{+\infty} + \left[w_i\tilde{h}_i^{(3)} + \tilde{T}_i^{(3)} + \int dv(\bar{\nu}\cdot v)^2\tilde{t}_i^{(3)}\right]_{-\infty}^{+\infty}$$
$$- \int_{-\infty}^{+\infty} dz w_i'\tilde{h}_i^{(3)} = 0. \tag{4.11}$$

By matching conditions we have, after summing on i, that the first two terms vanish because of the continuity of $\bar{\nu}\cdot\hat{u}$ and $\bar{\nu}\cdot\hat{u}^{(2)}$.

From the matching condition for $\tilde{f}_i^{(3)}$ we deduce

$$\sum_{i=1}^{2}\big[w_i\tilde{h}_i^{(3)} + \tilde{T}_i^{(3)}\big] \to \sum_{i=1}^{2}\big(\bar{\rho}_i\hat{h}_i^{(3)} + \bar{\rho}\hat{T}^{(3)}\big)^{\pm} + \sum_{i=1}^{2} d^{(1)}\bar{\nu}\cdot\nabla_r\big(\bar{\rho}_i\hat{h}_i^{(2)} + \bar{\rho}\hat{T}^{(2)}\big)^{\pm}. \tag{4.12}$$

$$\sum_{i=1}^{2}\int dv(\bar{\nu}\cdot v)^2\tilde{t}_i^{(3)} \to \sum_{i=1}^{2}\left(\int dv(\bar{\nu}\cdot v)^2\hat{t}_i^{(3)}\right)^{\pm}$$
$$+ d^{(1)}\bar{\nu}\cdot\nabla_r\left(\sum_{i=1}^{2}\int dv(\bar{\nu}\cdot v)^2\hat{t}_i^{(2)}\right)^{\pm} \tag{4.13}$$

where $\hat{t}_i^{(3)}$ is the projection of $\hat{f}_i^{(3)}$ on the orthogonal to the invariant space. Equation (3.37) shows that the parenthesis in the last term of (4.13) is equal to $\frac{1}{2}|\hat{u}|^2 - \eta\nu\cdot\nabla_r(\hat{u}\cdot\bar{\nu})$. The function $\bar{\rho}_i\hat{h}_i^{(2)} + \hat{T}_i^{(2)} + \frac{1}{2}|\hat{u}|^2$ is the pressure p. Moreover, after a computation based on (4.2) and the explicit expressions of $\hat{\gamma}_+^{(i)}$, $i = 1, 2$, using the continuity of $\nu\cdot\nabla_r\hat{u}^{(2)}$ we get

$$\left[\sum_{i=1}^{2}\left(\int dv(\bar{\nu}\cdot v)^2\hat{t}_i^{(3)}\right)\right]_-^+ = [\omega]_-^+.$$

To compute the last term in (4.11) we use the compatibility condition for the equation

$$\mathcal{Q}\tilde{\rho}_i^{(3)} = \tilde{h}_i^{(3)} - K^{(1)} \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)w_j(z')$$

where we used the definition of $\tilde{h}_i^{(3)}$ and the form of $\tilde{g}_i^{(3)}$ given in Appendix A. We recall that $K^{(1)}$ is $\Delta_r d^{(1)}$ and hence is not known at this stage. We have

$$\sum_{i=1}^{2} \int dz \tilde{h}_i^{(3)} w_i' = K^{(1)} \sum_{i=1}^{2} \int dz dz' w_i'(z)(z'-z)\tilde{U}(|z'-z|)w_j(z') = K^{(1)}\sigma.$$

Putting everything together and replacing in (4.11) we have that

$$\left[\sum_{i=1}^{2} \left(\bar{\rho}_i \hat{h}_i^{(3)} + \bar{\rho}\hat{T}^{(3)}\right) + \omega\right]_-^+ = K^{(1)}\sigma - d^{(1)}[\bar{\nu}\cdot\nabla_r p]^{\pm} + \eta d^{(1)}\left[\nu\cdot\nabla_r(\nu\cdot\nabla_r(\nu\cdot\hat{u}))\right]_-^+ .$$

Since by (4.6) the quantity $\sum_{i=1}^{2}(\bar{\rho}_i \hat{h}_i^{(3)} + \bar{\rho}\hat{T}^{(3)}) + \omega$ is the third correction to the pressure, the previous equation gives the wanted jump boundary condition for the pressure.

In summary, we got the following set of coupled equations for $\hat{u}^{(2)}$, $P^{(3)}$, $d^{(1)}$

$$\begin{aligned}
\partial_t \hat{u}^{(2)} + u\cdot\nabla_r \hat{u}^{(2)} + \nabla_r P^{(3)} &= \eta\Delta_r \hat{u}^{(2)} + N \\
\mathrm{div}\hat{u}^{(2)} &= 0, \\
V^{(1)} &= -\bar{\nu}\cdot\hat{u}^{(2)} + \nu^{(1)}\cdot\hat{u} - d^{(1)}\bar{\nu}\cdot\nabla_r\hat{u}, \\
\left[p^{(3)}\right]_-^+ &= K^{(1)}\sigma - d^{(1)}[\bar{\nu}\cdot\nabla_r p]^{\pm} + \eta d^{(1)}\left[\nu\cdot\nabla_r(\nu\cdot\nabla_r(\nu\cdot\hat{u}))\right]_,^+ \\
\nabla_r d^{(0)}\cdot\nabla_r d^{(1)} &= 0,
\end{aligned} \tag{4.14}$$

with $\hat{u}^{(2)}$ continuous and initial datum $\hat{u}^{(2)}(r,0)$ and $d^{(1)}(r,0) = 0, r \in \bar{\Gamma}_0$.

These are linearized equations in two different ways: the transport term is linear and the boundary $\bar{\Gamma}_t$ is given. In the last respect, they are similar to the linearized Hele-Shaw problem in.[1] The strategy to prove the existence of a solution is then analogous to the one in[1]: find the value of $d^{(1)}$ on $\bar{\Gamma}_t$ by using the first four equations and then use the last one to find $d^{(1)}$ in $\mathcal{N}^0(m)$.

We conclude this section by remarking that one can go on constructing the terms of any order in the expansion along the same lines and these terms will solve linearized initial boundary problems on a given surface. Only the first order term is solution of an initial free boundary problem. This is characteristic of the Hilbert expansion. One can also give different procedures such as Chapmann-Enskog expansion so that the terms at any order solve non linear problems. An algorithm to construct approximate solutions to Cahn-Hilliard equation which gives free boundary problems at any order has been given recently in Ref. (10).

5. VLASOV-NAVIER-STOKES EQUATIONS

In this section we consider the Vlasov-Navier-Stokes equations (VNS) derived in Ref. (2) from the VB Eq. (1.6). We show that in the sharp interface limit these equations reduce to the same free boundary problem (1.11)–(1.13) we found in the previous section.

We start by rewriting the equations in term of dimensionless quantities. We introduce the typical velocity $\bar{u}$, the length of the box L, the range of the Vlasov potential ℓ, the time scale τ, the sound speed c, $\bar{T} = c^2$ the typical temperature, $\bar{V}$ the intensity of the potential. In terms of this scales we can define the numbers $\mathcal{R}$ (Reynolds number), $\mathcal{M}$ (Mach number), $\mathcal{P}$ (Prandlt number), $\mathcal{A}$ and $\mathcal{P}'$

$$\mathcal{R} = \frac{\bar{u}L}{\bar{\eta}}, \qquad \mathcal{M} = \frac{\bar{u}}{c}, \qquad \mathcal{P} = \frac{\bar{\eta}}{\bar{k}}, \qquad \mathcal{P}' = \frac{\bar{D}}{\bar{\eta}}, \qquad \mathcal{A} = \frac{\bar{V}}{c^2}, \qquad \delta = \frac{\ell}{L},$$

where $\bar{\eta}$, $\bar{D}$ and $\bar{\kappa}$ are the characteristic strength of the viscosity, the diffusion coefficients and the heat conductivity. Then the VNS equations can be put in dimensionless form as

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0,\\
&\partial_t \varphi + \nabla \cdot (\varphi u) = \frac{\mathcal{P}'}{\mathcal{R}} \nabla \cdot (DQ),\\
&\rho D_t u + \frac{1}{\mathcal{M}^2}\nabla P - \frac{\mathcal{A}}{\mathcal{M}^2}\rho G^\delta * \rho + \frac{1}{\mathcal{M}^2}\varphi G^\delta * \varphi = -\frac{1}{\mathcal{R}}\nabla q,\\
&\frac{3}{2}\rho D_t T + P\nabla \cdot u = \frac{1}{\mathcal{R}\,\mathcal{P}}\nabla(\kappa \nabla T) - \frac{\mathcal{M}^2}{\mathcal{R}} q : \nabla u - \frac{\mathcal{P}'}{\mathcal{R}} G^\delta * \phi \cdot DQ\,. \qquad (5.1)
\end{aligned}$$

The notation is:

$$\begin{aligned}
D_t &:= \partial_t + u \cdot \nabla,\\
q &:= -\eta\big(\nabla u + (\nabla u)^\dagger - \frac{2}{3}\mathbb{I}\,\nabla \cdot u\big),\\
Q &:= \nabla \frac{\varphi}{\rho} + \frac{\mathcal{A}}{(\rho)^2 T}((\rho)^2 - (\phi)^2) G^\delta * \phi.
\end{aligned}$$

$(\nabla u)^\dagger$ is the adjoint of the matrix ∇u, $q : \nabla u = \mathrm{Tr}\,(q\nabla u)$, $\mathbb{I}$ is the unit matrix, $\rho = \rho_1 + \rho_2$ is the total density, $\varphi = \rho_1 - \rho_2$ is the concentration, $P = \rho T$ is the pressure, $G^\delta = \nabla U^\delta$, $U^\delta = \delta^{-d} U(\delta^{-1} r)$.

It is well known that the compressible Navier-Stokes equations reduce in the limit $\mathcal{M} \to 0$, taking fixed all the other dimensionless parameters, to the incompressible Navier-Stokes equations. We want to consider this limit here and at the same time the limit of sharp interface, that means $\delta \to 0$. We call $\mathcal{M} = \varepsilon$

and choose $\delta = \mathcal{M}^2$, keeping fixed all the other parameters (will be put equal to 1). The result is

$$\partial_t \rho^\varepsilon + \nabla \cdot [\rho^\varepsilon u^\varepsilon] = 0,$$

$$\partial_t \varphi^\varepsilon + \nabla \cdot (\varphi^\varepsilon u^\varepsilon) = \nabla \cdot (DQ^\varepsilon),$$

$$\rho D_t u^\varepsilon + \frac{1}{\varepsilon^2} \nabla P^\varepsilon - \frac{1}{\varepsilon^2} \rho^\varepsilon G^\varepsilon * \rho^\varepsilon + \frac{1}{\varepsilon^2} \varphi^\varepsilon G^\varepsilon * \varphi^\varepsilon = -\nabla q^\varepsilon,$$
$$\frac{3}{2} \rho^\varepsilon D_t T^\varepsilon + P \nabla \cdot u^\varepsilon = \nabla(\kappa \nabla T^\varepsilon) - \varepsilon^2 q^\varepsilon : \nabla u^\varepsilon - \mathbf{G}^\varepsilon * \varphi^\varepsilon \cdot DQ^\varepsilon. \qquad (5.2)$$

where $S^\varepsilon(x, t) = S^\varepsilon(\varepsilon^{-1} x, \varepsilon^{-2} t)$, with $S(x, t) = \{\rho, \phi, u, T\}(x, t)$.

We want to study the solutions of these equations in the limit $\varepsilon \to 0$. The front solutions (2.3), after the appropriate scaling, are stationary solution of the VNS equations. We consider as before an initial situation in which an interface Γ^0 is present and choose as initial datum for our system (2.2). Moreover, for sake of simplicity, we assume that at time zero $T^{(1)}$ and $\rho^{(1)}$ in the bulk are zero.

We are keeping all the notations of Section 2.

We look for a solutions in terms of an expansion in the parameter ε

$$\rho^\varepsilon = \sum_{s=0}^{\infty} \varepsilon^s \big[\hat{\rho}^{(s)}(r, t) + \tilde{\rho}^{(s)}(z, r, t)\big], \quad \varphi^\varepsilon = \sum_{s=0}^{\infty} \varepsilon^s \big[\hat{\varphi}^{(s)}(r, t) + \tilde{\varphi}^{(s)}(z, r, t)\big]$$
$$T^\varepsilon = \sum_{s=0}^{\infty} \varepsilon^s \big[\hat{T}^{(s)}(r, t) + \tilde{T}^{(s)}(z, r, t)\big], \quad u^\varepsilon = \sum_{s=0}^{\infty} \varepsilon^s \big[\hat{u}^{(s)}(r, t) + \tilde{u}^{(s)}(z, r, t)\big]$$

where the functions with tilde are fast functions depending on $\varepsilon^{-2} d(r, t)$ defined only near the interface at time t, Γ_t, while the other functions are defined in the complement.

We match the expansion in the bulk and near the interface of $S^\varepsilon = (\rho^\varepsilon, \varphi^\varepsilon, u^\varepsilon, T^\varepsilon)$ by requiring that

$$\lim_{z \to \pm\infty} \tilde{S}^{(0)}(z, r, t) = \big(\hat{S}^{(0)}\big)^{\pm}(r, t), \quad \lim_{z \to \pm\infty} \tilde{S}^{(1)}(z, r, t) = \big(\hat{S}^{(1)}\big)^{\pm}(r, t) \qquad (5.3)$$

$$\Big(\big(\hat{S}^{(2)}\big)^{\pm} + \bar{\nu} \cdot \big(\nabla_r \hat{S}^{(1)}\big)^{\pm} d^{(1)} + z \bar{\nu} \cdot \nabla_r \big(\hat{S}^{(0)}\big)^{\pm}\Big)(r, t)$$
$$= \tilde{S}^{(2)}(z, r, t) + o(1) \quad \text{as } z \to \pm\infty , \qquad (5.4)$$

...

Outer expansion

We will be very sketchy here since the incompressible limit for the Navier-Stokes equations is a well established topic.[19] From the equation for the velocity field we see that necessarily

$$\nabla \hat{P}^{\varepsilon} - \hat{\rho}^{\varepsilon} \hat{G}^{\varepsilon} * \hat{\rho}^{\varepsilon} + \hat{\varphi}^{\varepsilon} \hat{G}^{\varepsilon} * \hat{\varphi}^{\varepsilon} = \mathcal{O}(\varepsilon^2)$$

To compute the expansion of the force terms we use a simple argument based on Taylor expansion (see Appendix A) for the functions appearing in the outer expansion

$$U^{\varepsilon} * h(r,t) = h(r,t)\hat{U} + \varepsilon^4 \Delta_r h(r,t)\bar{U} + O(\varepsilon^6)$$

with $\hat{U} = \int U(r)dr, \quad \bar{U} = \int r^2 U(r)dr$. Hence, the previous condition gives

$$\nabla \hat{P}^{(0)} - \hat{\rho}^{(0)}\hat{U}\nabla\hat{\rho}^{(0)} + \hat{\varphi}^{(0)}\hat{U}\nabla\hat{\varphi}^{(0)} = 0$$

$$\nabla \hat{P}^{(1)} - \hat{\rho}^{(0)}\hat{U}\nabla\hat{\rho}^{(1)} + \hat{\varphi}^{(0)}\hat{U}\nabla\hat{\varphi}^{(1)} - \hat{\rho}^{(1)}\hat{U}\nabla\hat{\rho}^{(0)} + \hat{\varphi}^{(1)}\hat{U}\nabla\hat{\varphi}^{(0)} = 0$$

By choosing at initial time $\hat{\rho}^{(0)}$ and $\hat{\varphi}^{(0)}$ independent of x (and remembering the choice $\hat{\rho}^{(1)} = \hat{T}^{(1)} = 0$) we see that the previous conditions can be satisfied at any time by choosing $\hat{\rho}^{(0)}$, $\hat{\varphi}^{(0)}$ equal to these constant values and $\hat{\rho}^{(1)} = \hat{T}^{(1)} = 0$ at any time. With this choice, the continuity equation at zero order gives div $\hat{u} = 0$ and by the equation for the concentration $\hat{\varphi}^{(0)}$ we also obtain that $\hat{Q}^{(0)}$ is constant. Finally, the equation for the temperature becomes

$$\hat{\rho}^{(0)}\big(\partial_t \hat{T}^{(0)} + \hat{u}^{(0)} \cdot \nabla \hat{T}^{(0)}\big) = \nabla\big(k\nabla \hat{T}^{(0)}\big)$$

which has as unique solution $\hat{T}^{(0)}$ equal to its constant value at time zero. The momentum equation at order zero is

$$\hat{\rho}^{(0)}\big(\partial_t \hat{u}^{(0)} + u^{(0)} \cdot \nabla \hat{u}^{(0)}\big) + \nabla p = \eta \triangle \hat{u}^{(0)}$$

with

$$\nabla p := \nabla \hat{P}^{(2)} - \hat{\rho}^{(0)}\hat{U}\nabla\hat{\rho}^{(2)} + \hat{\varphi}^{(0)}\hat{U}\nabla\hat{\varphi}^{(2)}. \tag{5.5}$$

Inner expansion

By using (2.6) we have that at the lowest order ε^{-4} the conditions are

$$\partial_z\big(D\partial_z \tilde{Q}^{(0)}\big) = 0 \tag{5.6}$$

$$\bar{\nu}\partial_z \tilde{P}^{(0)} - \tilde{\rho}^{(0)}\tilde{G}^0 * \tilde{\rho}^{(0)} + \tilde{\varphi}^{(0)}\tilde{G}^0 * \tilde{\varphi}^{(0)} = \eta\left[\partial_z^2 \tilde{u}^{(0)} + \frac{1}{3}\bar{\nu}\partial_z^2\big(\tilde{u}^{(0)} \cdot \bar{\nu}\big)\right] \tag{5.7}$$

$$\partial_z\big(k\partial_z \tilde{T}^{(0)}\big) - \tilde{G}^{(0)} * \tilde{\varphi}^{(0)} \cdot D\tilde{Q}^{(0)} = 0 \tag{5.8}$$

where

$$\tilde{G}^0 * h = -\bar{\nu}\int dz'\partial_z \tilde{U}(|z - z'|)h(z')$$

$$\tilde{Q}^{(0)} = \partial_z \frac{\tilde{\varphi}^{(0)}}{\tilde{\rho}^{(0)}} + \frac{1}{\big(\tilde{\rho}^{(0)}\big)^2 \tilde{T}^{(0)}}\Big(\big(\tilde{\rho}^{(0)}\big)^2 - \big(\tilde{\varphi}^{(0)}\big)^2\Big)\tilde{G}^0 * \tilde{\varphi}^{(0)}.$$

We are using the notation: $\tilde{Q}^\varepsilon = \sum_{s=0}^\infty \varepsilon^{s-2}\tilde{Q}^{(s)}$, $\tilde{G}^\varepsilon = \sum_{s=0}^\infty \varepsilon^{s-2}\tilde{G}^{(s)}$.

The conditions at infinity for the densities imply the vanishing of $\tilde{Q}^{(0)}$ and $\partial_z\tilde{Q}^{(0)}$ at infinity. Hence, from the first equation we obtain $\tilde{Q}^{(0)} = 0$ identically. Then the third equation together with the matching conditions implies that $\tilde{T}^{(0)}$ is constant. From the second equation, since $\bar{\nu}\cdot\bar{\tau} = 0$, for any τ in the tangent plane to $\bar{\Gamma}_t$, we have that $\tilde{u}^{(0)}\cdot\bar{\tau}$ satisfies

$$\partial_z^2\big(\bar{\tau}\cdot\tilde{u}^{(0)}\big) = 0,$$

with the boundary condition $\bar{\tau}\cdot\tilde{u}^{(0)}(\pm\infty) = \bar{\tau}\cdot(\hat{u}^{(0)})^{\pm}$. This implies also $\partial_z(\bar{\tau}\cdot\tilde{u}^{(0)}) = 0$ at $\pm\infty$ and since the derivative has to be constant we have also $\partial_z(\bar{\tau}\cdot\tilde{u}^{(0)}) = 0$ which in turn implies that $\tilde{u}^{(0)}\cdot\bar{\tau}$ is a constant.

We assume that the first term of the expansion for the density is the front. The equations for the front are $\tilde{Q}^{(0)} = 0$ and

$$\bar{\nu}\partial_z\tilde{P}^{(0)} - \tilde{\rho}^{(0)}\tilde{G}^0 * \tilde{\rho}^{(0)} + \tilde{\varphi}^{(0)}\tilde{G}^0 * \tilde{\varphi}^{(0)} = 0$$

since $\tilde{T}^{(0)}$ is a constant. As a consequence, we have that

$$\partial_z^2\big(\nu\cdot\tilde{u}^{(0)}\big) = 0. \tag{5.9}$$

This equation together with the matching conditions give that $\tilde{u}^{(0)}\cdot\bar{\nu}$ is independent of z.

At the order ε^{-3} we have similar conditions:

$$\partial_z\big(D\partial_z\tilde{Q}^{(1)}\big) = 0 \tag{5.10}$$

$$\bar{\nu}\partial_z\tilde{P}^{(1)} - \tilde{\rho}^{(0)}G^0 * \tilde{\rho}^{(1)} + \tilde{\rho}^{(1)}G^0 * \tilde{\rho}^{(0)} + \tilde{\varphi}^{(1)}\tilde{G}^0 * \tilde{\varphi}_0 + \tilde{\varphi}_0\tilde{G}^0 * \tilde{\varphi}^{(1)}$$
$$= \eta\left[\partial_z^2\tilde{u}^{(1)} + \frac{1}{3}\bar{\nu}\partial_z^2\big(\tilde{u}^{(1)}\cdot\bar{\nu}\big)\right] \tag{5.11}$$

$$\partial_z\big(\kappa\partial_z\tilde{T}^{(1)}\big) - \tilde{G}^{(0)} * \tilde{\varphi}^{(0)}\cdot D\tilde{Q}^{(1)} - \tilde{G}^{(0)} * \tilde{\varphi}^{(1)}\cdot D\tilde{Q}_0 = 0. \tag{5.12}$$

We notice that $\tilde{T}^{(1)} = \tilde{\varphi}^{(1)} = \tilde{\rho}^{(1)} = 0$ satisfy Eqs. (5.10) and (5.12) and also the matching conditions. This choice implies $\partial_z^2\tilde{u}^{(1)} = 0$.

At the order ε^{-2}:

$$V\partial_z\tilde{\rho}^{(0)} + \partial_z\big(\tilde{\rho}^{(0)}\bar{\nu}\cdot\tilde{u}^{(0)}\big) = 0 \tag{5.13}$$

$$V\partial_z\tilde{\varphi}^{(0)} + \partial_z\big[\tilde{\varphi}^{(0)}\bar{\nu}\cdot\tilde{u}^{(0)}\big] = \bar{K}D\partial_z\tilde{Q}^{(0)} + D\partial_z^2\tilde{Q}^{(2)}, \tag{5.14}$$

$$0 = \frac{4}{3}\big(\partial_z\big(\tilde{u}^{(0)}\cdot\bar{\nu}\big)\big)^2 + \big(\partial_z\big(\tilde{u}^{(0)}\cdot\bar{\tau}\big)\big)^2 + \partial_z\big(\kappa\partial_z\tilde{T}^{(2)}\big) - \tilde{G}(0) * \tilde{\varphi}^{(0)}\cdot D\tilde{Q}^{(2)} \tag{5.15}$$

where $\bar{K}$ is the curvature of $\bar{\Gamma}_t$ (see (2.6)).

Equation (5.13) implies, since $\bar{\nu} \cdot \tilde{u}^{(0)}$ is independent of z, that

$$V = -\bar{\nu} \cdot \tilde{u}^{(0)} . \tag{5.16}$$

Replacing in (5.14) we get $\partial_z^2 \tilde{Q}^{(2)} = 0$.

Moreover,

$$\begin{aligned} &V\partial_z \tilde{u}^{(0)} + \big(\tilde{u}^{(0)} \cdot \bar{\nu}\big)\partial_z u^{(0)} + \bar{\nu}\partial_z \tilde{P}^{(2)} - \tilde{F}^{(2)} - \tilde{\rho}^{(0)} \frac{\bar{K}}{2} \bar{\nu}\partial_z \\ &\int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)\tilde{\rho}^{(0)}(z') \\ &+\tilde{\varphi}^{(0)} \frac{K}{2} \bar{\nu}\partial_z \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)\tilde{\varphi}^{(0)}(z') = \eta \left[\partial_z^2 \tilde{u}^{(2)} + \frac{1}{3}\bar{\nu}\partial_z^2 \big(\tilde{u}^{(2)} \cdot \bar{\nu}\big)\right] \end{aligned} \tag{5.17}$$

where

$$\tilde{F}^{(2)} = -\tilde{\rho}^{(0)} G^0 * \tilde{\rho}^{(2)} - \tilde{\rho}^{(2)} G^0 * \tilde{\rho}^{(0)} + \tilde{\varphi}^{(2)} \tilde{G}^0 * \tilde{\varphi}^{(0)} + \tilde{\varphi}^{(0)} \tilde{G}^0 * \tilde{\varphi}^{(2)}.$$

We have used

$$U^\varepsilon * \tilde{h} = \tilde{U} * \tilde{h} + \varepsilon^2 \frac{\bar{K}}{2} \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)\tilde{h}(z') + O(\varepsilon^4). \tag{5.18}$$

The first consequence of this equation is

$$\partial_z^2 \big(\tilde{u}^{(2)} \cdot \bar{\tau}\big) = 0. \tag{5.19}$$

Taking into account this relation and (5.16) we write (5.17) as

$$\begin{aligned} &\bar{\nu}\partial_z \tilde{P}^{(2)} - \tilde{F}^{(2)} - \bar{\nu}\tilde{\rho}^{(0)} \frac{\bar{K}}{2} \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)\tilde{\partial}_z \rho^{(0)}(z') \\ &\quad + \bar{\nu}\tilde{\varphi}^{(0)} \frac{\bar{K}}{2} \int_{\mathbb{R}} dz'(z'-z)\tilde{U}(|z'-z|)\partial_z \tilde{\varphi}^{(0)}(z') = \eta\bar{\nu}\frac{4}{3}\partial_z^2 (\tilde{u}^{(2)} \cdot \bar{\nu}). \end{aligned} \tag{5.20}$$

Since $\int dz \tilde{F}^{(2)} = \bar{\nu} \int dz \partial_z [\tilde{\rho}^{(0)} \tilde{U} * \tilde{\rho}^{(2)} - \tilde{\varphi}^{(2)} \tilde{U} * \tilde{\varphi}^{(0)}]$ by integrating over z we get

$$\begin{aligned} \big[\tilde{P}^{(2)}\big]_{-\infty}^{+\infty} = &-\Big[\int_{\mathbb{R}} dz' \tilde{U}(|z'-z|)\big[-\tilde{\rho}^{(0)}\tilde{\rho}^{(2)} + \tilde{\varphi}_0 \tilde{\varphi}^{(2)}\big]_{-\infty}^{+\infty} + \big[\partial_z (\tilde{u}^{(2)} \cdot \bar{\nu})\big]_{-\infty}^{+\infty} \\ &+ \frac{\bar{K}}{2} \int_{\mathbb{R}^2} dz dz'(z'-z)\tilde{U}(|z'-z|)[\tilde{\rho}^{(0)}(z)\rho^{(0)'}(z') - \tilde{\varphi}^{(0)}(z)\tilde{\varphi}^{(0)'}(z')]. \end{aligned} \tag{5.21}$$

The last term is equal to $\bar{K}\sigma$ (see Appendix B). Using the matching conditions we have

$$\big[\tilde{P}^{(2)}\big]_{-\infty}^{+\infty} = \big[\hat{P}^{(2)}\big]_{-}^{+}, \quad \big[\partial_z(\tilde{u}^{(2)}\cdot\bar{\nu})\big]_{-\infty}^{+\infty} = \big[\bar{\nu}\cdot\nabla(\hat{u}^{(0)}\cdot\bar{\nu})\big]_{-}^{+}$$
$$\left[\int_{\mathbb{R}} dz'\tilde{U}(|z'-z|)\big[\tilde{\rho}^{(0)}\tilde{\rho}^{(2)} - \tilde{\varphi}^{(0)}\tilde{\varphi}^{(2)}\big]\right]_{-\infty}^{+\infty} = \hat{U}\big[\hat{\rho}^{(0)}\hat{\rho}^{(2)} - \hat{\varphi}^{(0)}\hat{\varphi}^{(2)}\big]_{-}^{+}.$$

The continuity of $\hat{u}^{(0)}$ gives obviously that $\big[\bar{\tau}\cdot\nabla(\hat{u}^{(0)}\cdot\bar{\tau})\big]_{-}^{+} = 0$. This and $\mathrm{div}\hat{u}^{(0)} = 0$ give also

$$\big[\bar{\nu}\cdot\nabla(\hat{u}^{(0)}\cdot\bar{\nu})\big]_{-}^{+} = 0.$$

Therefore,

$$\big[\hat{P}^{(2)}\big]_{-}^{+} - \hat{U}\left[\hat{\rho}^{(0)}\hat{\rho}^{(2)} - \hat{\varphi}^{(0)}\hat{\varphi}^{(2)}\right]_{-}^{+} = \bar{K}\sigma.$$

The pressure p in the incompressible Navier-Stokes equation is given by (5.5) as

$$p = \hat{P}^{(2)} - \hat{U}\big(\hat{\rho}^{(0)}\hat{\rho}^{(2)} - \hat{\varphi}^{(0)}\hat{\varphi}^{(2)}\big).$$

Hence,

$$[p]_{-}^{+} = \bar{K}\sigma$$

In conclusion, the inner expansion provide the boundary conditions on $\bar{\Gamma}_t$ to be added to the hydrodynamical equations obtained from the outer expansion.

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A.1. FORCE TERMS

For a slowly varying function $h(r, t)$ we have that

$$U^{\varepsilon} * h(r,t) = \int_{\mathbb{R}^3} \varepsilon^{-6}U(\varepsilon^{-2}|r-r'|)h(r',t)dr'$$
$$= \int_{\mathbb{R}^3} U(|q-q'|)[h(\varepsilon^2 q',t) - h(\varepsilon^2 q,t)]dq' + h(r,t)\int_{\mathbb{R}^3} U(|q-q'|)dq'$$

$$= \int_{\mathbb{R}^3} U(|q-q'|)\Big[\varepsilon^2(q-q')\cdot\nabla_r h(\varepsilon^2 q, t)$$

$$+ \frac{\varepsilon^4}{2}\sum_{i,j}(q-q')_i(q-q')_i\partial_i\partial_j h(\varepsilon^2 q, t) + O(\varepsilon^6)\Big]dq' + h(r,t)\hat{U}$$

$$= h(r,t)\hat{U} + \varepsilon^4\Delta_r h(r,t)\bar{U} + O(\varepsilon^6) \tag{A.1}$$

where $\hat{U} = \int U(r)dr, \quad \bar{U} = \int r^2 U(r)dr$.

To compute the expansion of $U^\varepsilon * h$ for a fast varying function $h(z, r, t)$ it is more convenient to use a local system of coordinates. Fix $d = 3$. For a given curve Γ and for any point $s \in \Gamma$ we choose a reference frame centered in s with the axes $1, 2$ along the directions of principal curvatures k_i and 3 in the direction of the normal. Consider two points r and r' and choose the reference frame centered in $s(r) : r = s(r) + \varepsilon^2 z\nu(r)$. We denote by y_i and y_i' the coordinates of $q = \varepsilon^{-2}r$ and $q' = \varepsilon^{-2}r'$ in this new frame. Then $y_1 = y_2 = 0$, $y_3 = z$ and $q_i' = A_{i\ell}y_\ell'$, with $A_{\cdot j}$ the component j of the normal and the tangents to the surface in r. We will use the notation $r'(\{y_i'\}) = \varepsilon^2 A_{i\ell}y_\ell'$ and $z'(\{y_i'\})$ which is given by ([17])

$$z' = y_3' + \textstyle\sum_{i=1,2}\frac{1}{2}\big[\varepsilon^2 k_i y_i'^2 - 2\varepsilon^4 k_i^2 y'^2_i y_3'\big] + O(\varepsilon^6), \tag{A.2}$$

where z' is defined by $r' = s(r') + \varepsilon^2\nu(r')z'$ and $h(z', r', t) = h(z'(\{y_i'\}), r'(\{y_i'\}), t)$. We have by Taylor expansion

$$(U^\varepsilon * h)(z, r, t) = \int_{\mathbb{R}^3} dy' U(|y-y'|)h(z'(\{y_i'\}), \varepsilon^2 q'(\{y_i'\}), t)$$

$$= \int_{\mathbb{R}^3} dy' U(|y-y'|)h(z'(0,0,y_3'), \varepsilon^2 q(0,0,y_3'), t)$$

$$+ \frac{1}{2}\sum_{i=1,2}\left(\tilde{U}_{1,i} * \frac{\partial^2\check{h}}{\partial y_i'^2}\right) + \frac{1}{4}\sum_{i=1,2}\left(\tilde{U}_{2,i} * \frac{\partial^4\check{h}}{\partial y_i'^4}\right) \tag{A.3}$$

$$+ \frac{1}{4}\sum_{i,j=1,2,i\neq j}\tilde{U}_{2,ij} * \frac{\partial^4\check{h}}{\partial y'^2_i y'^2_j} + N,$$

where

$$\check{h}\big(y_1', y_2', y_3', t\big) := h\big(z'\big(\{y_i'\}\big), \varepsilon^2 q'(\{y_i'\}), t\big)$$

$$\tilde{U}_{s,i}(|q_3 - q_3'|) = \int_{\mathbb{R}^2} dq' U\left(\sqrt{|y_3 - y_3'|^2 + |y_1'|^2 + |y_2'|^2}\right)|y_i'|^{2s}$$

We have

$$\frac{\partial \check{h}}{\partial y'_i} = \varepsilon^2 \sum_j \bar{\nabla}_j h A_{ji} + \frac{\partial h}{\partial z}\frac{\partial z'}{\partial y'_i}$$

$$\frac{\partial^2 \check{h}}{\partial {y'}_k^2} = \varepsilon^4 \sum_{j\ell} A_{jk} A_{\ell k} \bar{\nabla}^2_{j\ell} h + \frac{\partial z'}{\partial y'_k}\left[\varepsilon^2 A_{jk} \bar{\nabla}_j \frac{\partial h}{\partial z} + \frac{\partial z'}{\partial y'_k}\frac{\partial^2 h}{\partial z^2}\right] + \frac{\partial h}{\partial z}\frac{\partial^2 z'}{\partial {y'}_k^2}\,. \tag{A.4}$$

We replace this expression in (A.3). The contribution of order ε^2 comes only from the last term in (A.4) because the contribution of the third term of this order is zero for $k \neq 3$. By using the relation between z' and y'_3 (A.2) we see that the last term equals to

$$\frac{\partial h}{\partial z}\big(0, 0, y'_3, \big)\big(\varepsilon^2 k_k - \varepsilon^4 2 k_k^2 y'_3\big).$$

Therefore, the second term in (A.3) gives at order ε^2

$$\int_{\mathbb{R}^3} dy' U(|y - y'|)\frac{\partial h}{\partial z}\big(0, 0, y'_3\big) \sum_{i=1}^{d-1} \frac{k_i^2 {y'}_i^2}{2} = \frac{K}{2}\int_{\mathbb{R}} dz'(z' - z)\tilde{U}(|z' - z|)h(z', r). \tag{A.5}$$

The equality is proven in Ref. (5).

To compute the contributions at different order in ε we go back to the specific curve Γ_t^ε and use the expansion $d^\varepsilon(r, t) = \sum_n \varepsilon^n d^{(n)}(r, t)$ which implies $k_i^\varepsilon = \sum_n \varepsilon^n k_i^{(n)}$ and $A_{ij}^\varepsilon = \sum_n \varepsilon^n A_{ij}^{(n)}$.

In conclusion,

$$\begin{aligned}(U^\varepsilon * h)(z, r) &= (\tilde{U} * h)(z, r) + \varepsilon^2 \frac{\bar{K}}{2} C(h) + \varepsilon^3 \frac{K^{(1)}}{2} C(h)\\ &+ \sum_{i=1,2}\Bigg[\varepsilon^4\Bigg(\tilde{U}_{1,i} * D_{1,i}(h) + \tilde{U}_{2,i} * D_{2,i}(h) + \sum_{j\neq i}\tilde{U}_{2,ij} * D_{2,ij}(h)\\ &+ \frac{K^{(2)}}{2} C(h)\Bigg) + O(\varepsilon^5)\Bigg] := (\tilde{U} * h)(z, r) + \sum_{n=1}^{2} \varepsilon^n B_n(h) + O(\varepsilon^5),\end{aligned} \tag{A.6}$$

where $\bar{K} := K^{(0)}$ and

$$D_{1,i}(h) = \frac{1}{2}\sum_{j\ell} A_{ji}^{(0)} A_{\ell i}^{(0)} \bar{\nabla}^2_{j\ell} h - \frac{\partial h}{\partial z} 2\left(k_i^{(0)}\right)^2 y'_3;\quad D_{2,i}(h) = \frac{3 - 6y'_3}{4}\big(k_i^{(0)}\big)^2 \frac{\partial^2 h}{\partial z^2};$$

$$C(h) = \int_{\mathbb{R}} dz'(z' - z)\tilde{U}(|z' - z|)h(z', r), \quad D_{2,ij}(h) = \frac{1}{4}k_i^{(0)}k_j^{(0)}\frac{\partial^2 h}{\partial z^2}.$$

We use the notation:

$$\left(U^\varepsilon * \hat{\rho}_j^\varepsilon\right)(r, t) = \sum_{n=0}^{N} \varepsilon^n \hat{g}_i^{(n)}, \quad \left(U^\varepsilon * \tilde{\rho}_j^\varepsilon\right)(r, t) = \sum_{n=0}^{N} \varepsilon^n \tilde{g}_i^{(n)} \tag{A.7}$$

We have that

$$\hat{g}_i^{(n)} = \hat{U}\nabla_r \hat{\rho}_i^{(n)}, \; n = 0, 1; \quad \hat{g}_i^{(2)} = \hat{U}\nabla_r \hat{\rho}_i^{(2)} + \bar{U}\Delta_r \hat{\rho}_i^{(0)} \tag{A.8}$$

$$\tilde{g}_i^{(n)} = \tilde{U} * \nabla_r \tilde{\rho}_i^{(n)}, \; n = 0, 1; \quad \tilde{g}_i^{(2)} = \tilde{U} * \nabla_r \tilde{\rho}_i^{(2)} + B_2 \tag{A.9}$$

The previous computations can be generalized to compute the terms $\hat{g}^{(k)}$, $\tilde{g}^{(k)}$ at any order.

B.1. SURFACE TENSION

The computation in this Appendix is taken form.[21] The surface tension for a planar interface can be defined as the difference between the grand canonical free energy (pressure) of an equilibrium state with the interface and a homogeneous one.[27] We call excess pressure this difference. The pressure for this model is

$$\mathcal{P}(n_1, n_2) = \int \mathrm{d}x p(n_1(x), n_2(x))$$

$$p(n_1, n_2) = T(n_1 \log n_1 + n_2 \log n_2) + \frac{1}{2}n_1 U * n_2 + \frac{1}{2}n_2 U * n_1 - \mu_1 n_1 - \mu_2 n_2.$$

Consider the system in a cylinder of base $(2L)^{d-1}$ and height M in presence of a planar interface dividing the cylinder in the half upper cylinder where the densities are ρ_1^+, ρ_2^+ and the half lower cylinder with densities ρ_1^-, ρ_2^-, where $\rho_1^\pm, \rho_2^\pm$ are the equilibrium values of the densities in the coexisting phases at temperature T. Then the excess pressure is given by[5]

$$\sigma = \lim_{L\to\infty} \frac{1}{(2L)^{d-1}} \lim_{M\to\infty} \int_{-L}^{L} \mathrm{d}y_1 \ldots \int_{-L}^{L} \mathrm{d}y_{d-1} \int_{-M}^{M} dy_d \left[p(w_1, w_2) - p\left(\rho_1^+, \rho_2^-\right)\right]$$

where $w_i(q)$ are the front solutions, smooth functions satisfying the equations

$$T \log w_i(q) + \int_{\mathbb{R}} \mathrm{d}q' \tilde{U}(|q - q'|) w_j(q') = C_i \tag{B.1}$$

where $\tilde{U}(q) = \int_{\mathbb{R}^2} \mathrm{d}y U(\sqrt{q^2 + y^2})$ and $C_i = \mu_i - T$ are constants determined by the conditions at infinity $\rho_i^\pm$. Notice that $f(\rho_1^+, \rho_2^-) = f(\rho_1^-, \rho_2^+)$ since $\rho_1^\pm = \rho_2^\mp$ and that $\mu_1 = \mu_2 = \mu$ in the coexisting region.

We rewrite the surface tension by using integration by part and the condition at infinity

$$\sigma = \int_{-\infty}^{+\infty} \mathrm{d}z\big[p(w_1, w_2) - p\big(n_1^+, n_2^-\big)\big] = -\int_{-\infty}^{+\infty} \mathrm{d}z z \frac{d}{dz} p(w_1(z), w_2(z)).$$

We have

$$\frac{d}{dz}p(w_1, w_2) = T\big[(\log w_1 + 1)w_1' + (\log w_2 + 1)w_2'\big] + \frac{1}{2}\big[w_1'\tilde{U} * w_2 + w_2'\tilde{U} * w_1 + w_1\tilde{U} * w_2' + w_2\tilde{U} * w_1'\big] - \mu\big(w_1' + w_2'\big).$$

By using (B.1) and $C_1 = C_2 = C$ we get

$$\frac{d}{dz}p(w_1, w_2) = \frac{1}{2}\big[- w_1'\tilde{U} * w_2 - w_2'\tilde{U} * w_1 + w_1\tilde{U} * w_2' + w_2\tilde{U} * w_1'\big] + (C + T)\big(w_1' + w_2'\big) - \mu\big(w_1' + w_2'\big)$$

and for the surface tension, by remembering that $C = \mu - T$,

$$-\frac{1}{2}\int \mathrm{d}z\mathrm{d}z'\, z \sum_{i\neq j} \big[- w_i'(z)\tilde{U}(z - z')w_j(z') + w_i(z)\tilde{U}(z - z')w_j'(z')\big]$$

In conclusion,

$$\sigma = \frac{1}{2}\int dzdz'(z - z') \sum_{i\neq j} \big[w_i'(z)\tilde{U}(z - z')w_j(z')\big].$$

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